ON THE RIGOROUS DERIVATION OF THE 3D CUBIC NONLINEAR SCHRÖDINGER EQUATION WITH A QUADRATIC TRAP

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ABSTRACT. We consider the dynamics of the 3D N-body Schrödinger equation in the presence of a quadratic trap. We assume the pair interaction potential is $N^{3\beta-1}V(N^{\beta}x)$. We justify the mean-field approximation and offer a rigorous derivation of the 3D cubic NLS with a quadratic trap. We establish the space-time bound conjectured by Klainerman and Machedon [30] for $\beta \in (0, 2/7]$ by adapting and simplifying an argument in Chen and Pavlović [7] which solves the problem for $\beta \in (0, 1/4)$ in the absence of a trap.

CONTENTS

1. INTRODUCTION

It is widely believed that the cubic nonlinear Schrödinger equation (NLS)

$$i\partial_t \phi = L\phi + |\phi|^2 \phi,$$

where L is the Laplacian $-\triangle$ or the Hermite operator $-\triangle + \omega^2 |x|^2$, describes the physical phenomenon of Bose-Einstein condensation (BEC). This belief is one of the main motivations for studying the cubic NLS. BEC is the phenomenon that particles of integer spin ("Bosons") occupy a macroscopic quantum state. This unusual state of matter was first predicted theoretically by Einstein for non-interacting particles. The first experimental observation of BEC in an interacting atomic gas did not occur until 1995 using laser cooling techniques [1, 14]. E. A. Cornell, W. Ketterle, and C. E. Wieman were awarded the 2001 Nobel Prize in Physics for observing BEC. Many similar successful experiments were performed later on [4, 13, 27, 37].

Let $t \in \mathbb{R}$ be the time variable and $\mathbf{x}_N = (x_1, x_2, ..., x_N) \in \mathbb{R}^{3N}$ be the position vector of N particles in $\mathbb{R}^3$. Then BEC naively means that the N-body wave function $\psi_N(t, \mathbf{x}_N)$ satisfies

$$\psi_N(t, \mathbf{x}_N) = \prod_{j=1}^{N} \phi(t, x_j) \tag{1.1}$$

up to a phase factor solely depending on t, for some one particle state ϕ. In other words, every particle is in the same quantum state. It is believed that the one particle wave function ϕ which models the condensate satisfies the cubic NLS. Gross [25, 26] and Pitaevskii [34] proposed such a description. However, the cubic NLS is a phenomenological mean field type equation and its validity needs to be established rigorously from

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the many body system which it is supposed to characterize. As a result, we investigate the procedure of laboratory experiments of BEC according to [1, 14].

Step A. Confine a large number of Bosons inside a trap e.g., the magnetic fields in [1, 14]. Cool it down so that the many body system reaches its ground state. It is expected that this ground state is a BEC state / factorized state. This step corresponds to the mathematical problem.

Problem 1. *Show that the ground state of the N-body Hamiltonian*

$$\sum_{j=1}^{N}\left(-\frac{1}{2}\triangle_{x_j}+\frac{\omega_0^2}{2}\left|x_j\right|^2\right)+\frac{1}{N}\sum_{1\leqslant i<j\leqslant N}N^{3\beta}V\left(N^{\beta}\left(x_i-x_j\right)\right)$$

is a factorized state.

We use the quadratic potential $|x|^2$ to represent the trap. This simplified yet reasonably general model is expected to capture the salient features of the actual trap: on the one hand the quadratic potential varies slowly, on the other hand it tends to ∞ as $|x|\to\infty$. In the physics literature, Lieb, Seiringer and Yngvason remarked in [31] that the confining potential is typically $\sim|x|^2$ in the available experiments. Mathematically speaking, the strongest trap we can deal with in the usual regularity setting of NLS is the quadratic trap since the work [39] by Yajima and Zhang points out that the ordinary Strichartz estimates start to fail as the trap exceeds quadratic.

Step B. Switch the trap in order to enable measurement or direct observation. It is assumed that such a shift of the confining potential is instant and does not destroy the BEC obtained from Step A. To be more precise about the word "switch": in [1, 14] the trap is removed, in [37] the initial magnetic trap is switched to an optical trap, in [4] the trap is enhanced, in [13] the trap is turned off in 2 spatial directions to generate a 2D Bose gas. Hence we have a different trap after the switch, in other words, the trapping potential becomes $\omega^2|x|^2$. The system is then time dependent unless $\omega=\omega_0$. Therefore, the factorized structure obtained in Step A must be preserved in time for the observation of BEC. Mathematically, this step stands for the following problem.

Problem 2. *Take the BEC state obtained in Step A. as initial datum, show that the solution to the many body Schrödinger equation*

$$i\partial_t\psi_N=\sum_{j=1}^{N}\left(-\frac{1}{2}\triangle_{x_j}+\omega^2\frac{\left|x_j\right|^2}{2}\right)\psi_N+\frac{1}{N}\sum_{1\leqslant i<j\leqslant N}N^{3\beta}V(N^{\beta}\left(x_i-x_j\right))\psi_N \tag{1.2}$$

is a BEC state / factorized state.

We first remark that neither of the problems listed above admits a factorized state solution. It is also unrealistic to solve the equations in Problems 1 and 2 for large N. Moreover, both problems are linear so that it is not clear how the cubic NLS arises from either problem. Therefore, in order to justify the statement that the cubic NLS depicts BEC, we have to show mathematically that, in an appropriate sense,

$$\psi_N(t,\mathbf{x}_N)\sim\prod_{j=1}^{N}\phi(t,x_j)\text{ as }N\to\infty$$

for some one particle state ϕ which solves a cubic NLS. However, when $\phi\neq\phi'$

$$\left\|\prod_{j=1}^{N}\phi(t,x_j)-\prod_{j=1}^{N}\phi'(t,x_j)\right\|_{L^2}^2\to 2\ as\ N\to\infty.$$

i.e. our desired limit (the BEC state) is not stable against small perturbations. One way to circumvent this difficulty is to use the concept of the k-particle marginal density $\gamma_N^{(k)}$ associated with ψ_N defined as

$$\gamma_N^{(k)}(t,\mathbf{x}_k;\mathbf{x}_k')=\int\psi_N(t,\mathbf{x}_k,\mathbf{x}_{N-k})\overline{\psi_N}(t,\mathbf{x}_k',\mathbf{x}_{N-k})d\mathbf{x}_{N-k},\ \ \mathbf{x}_k,\mathbf{x}_k'\in\mathbb{R}^{3k} \tag{1.3}$$

and show that

$$\gamma_N^{(k)}(t,\mathbf{x}_k;\mathbf{x}_k')\sim\prod_{j=1}^{k}\phi(t,x_j)\bar{\phi}(t,x_j')\text{ as }N\to\infty.$$

Penrose and Onsager [33] suggested such a formulation. Another approach is to add a second order correction to the mean field approximation. See [11, 23, 24].

For Problem 1, Lieb, Seiringer, Solovej and Yngvason showed that the ground state of the Hamiltonian exhibits complete BEC in [32], provided that the trapping potential $V_{trap}(x)$ satisfies $\inf_{|x|>R}V_{trap}(x)\to\infty$

for $R \to \infty$ and the interaction potential is spherically symmetric. To be more precise, let $\psi_{N,0}$ be the ground state, then

$$\gamma_{N,0}^{(1)} \to |\phi_{GP}\rangle \langle \phi_{GP}| \text{ as } N \to \infty,$$

where $\gamma_{N,0}^{(1)}$ is the corresponding one particle marginal density defined via formula 1.3 and ϕ_{GP} is the minimizer of the Gross-Pitaevskii energy functional

$$\int (|\nabla\phi|^2 + V_{trap}(x) |\phi|^2 + 4\pi a_0 |\phi|^4)dx.$$

So far, there has not been any work regarding the Hamiltonian evolution in Step B in the case when $\omega \neq 0$. Motivated by the above considerations, we aim to investigate the evolution of a many-body Boson system with a quadratic trap and study the dynamics after the switch of the trap. We derive rigorously the 3D cubic NLS with a quadratic trap from the $N-$body linear equation 1.2. To be specific, we establish the following theorem in this paper.

Theorem 1. *(Main Theorem) Let $\left\{\gamma_N^{(k)}\right\}$ be the family of marginal densities associated with ψ_N, the solution of the N-body Schrödinger equation 1.2 for some $\beta \in \left(0, \frac{2}{7}\right]$. Assume that the pair interaction V is a nonnegative $L^1(\mathbb{R}^3) \cap H^2(\mathbb{R}^3) \cap W^{2,\infty}(\mathbb{R}^3)$ spherically symmetric function. Moreover, suppose the initial datum of equation 1.2 verifies the following the conditions:*

(a) the initial datum is normalized i.e.

$$\|\psi_N(0)\|_{L^2} = 1,$$

(b) the initial datum is asymptotically factorized

$$\lim_{N\to\infty} \operatorname{Tr} \left|\gamma_N^{(1)}(0, x_1; x_1') - \phi_0(x_1)\overline{\phi_0}(x_1')\right| = 0,$$

for some one particle wave function ϕ_0.

(c) the initial datum has bounded energy per particle i.e.

$$\sup_N \frac{1}{N} \langle \psi_N(0), H_N \psi_N(0)\rangle < \infty,$$

where the Hamiltonian H_N is

$$H_N = \sum_{j=1}^{N} \left(-\frac{1}{2}\triangle_{x_j} + \omega^2 \frac{|x_j|^2}{2}\right) + \frac{1}{N} \sum_{1\leqslant i<j\leqslant N} N^{3\beta} V(N^\beta (x_i - x_j)).$$

Then $\forall t \geqslant 0$, $\forall k \geqslant 1$, we have the convergence in the trace norm that

$$\lim_{N\to\infty} \operatorname{Tr} \left|\gamma_N^{(k)}(t, \mathbf{x}_k; \mathbf{x}_k') - \prod_{j=1}^{k} \phi(t, x_j)\overline{\phi}(t, x_j')\right| = 0,$$

where $\phi(t,x)$ is the solution to the 3D cubic NLS with a quadratic trap

$$\begin{aligned} i\partial_t\phi &= \left(-\frac{1}{2}\triangle_x + \omega^2\frac{|x|^2}{2}\right)\phi + b_0 |\phi|^2 \phi \text{ in } \mathbb{R}^{3+1} \\ \phi(0,x) &= \phi_0(x) \end{aligned} \tag{1.4}$$

and the coupling constant $b_0 = \int_{\mathbb{R}^3} V(x)dx$.

For the $\omega = 0$ case, the approach which uses the marginal densities $\left\{\gamma_N^{(k)}\right\}$ for the dynamics problem has been proven to be successful in the fundamental papers [15, 16, 17, 18, 19, 20, 21] by Elgart, Erdös, Schlein, and Yau. As pointed out in [21], their work corresponds to the evolution after the removal of the traps. Motivated by a kinetic formulation of Spohn [36], their program consists of two principal parts: in one part, they prove that an appropriate limit of the sequence $\left\{\gamma_N^{(k)}\right\}$ as $N \to \infty$ solves the Gross-Pitaevskii hierarchy

$$\left(i\partial_t + \frac{1}{2}\triangle_{\mathbf{x}_k} - \frac{1}{2}\triangle_{\mathbf{x}_k'}\right)\gamma^{(k)} = b_0 \sum_{j=1}^{k} B_{j,k+1}\left(\gamma^{(k+1)}\right),\ k = 1, ..., n, ... \tag{1.5}$$

where $B_{j,k+1} = B_{j,k+1}^1 - B_{j,k+1}^2$ and

$$B_{j,k+1}^1\left(\gamma^{(k+1)}\right)(t, \mathbf{x}_k; \mathbf{x}_k') = \int\int \delta(x_j - x_{k+1})\delta(x_{k+1} - x_{k+1}')\gamma^{(k+1)}(t, \mathbf{x}_{k+1}; \mathbf{x}_{k+1}')dx_{k+1}dx_{k+1}',$$

$$B_{j,k+1}^2\left(\gamma^{(k+1)}\right)(t, \mathbf{x}_k; \mathbf{x}_k') = \int\int \delta(x_j' - x_{k+1})\delta(x_{k+1} - x_{k+1}')\gamma^{(k+1)}(t, \mathbf{x}_{k+1}; \mathbf{x}_{k+1}')dx_{k+1}dx_{k+1}';$$

in another part, they show that hierarchy 1.5 has a unique solution which is therefore a completely factorized state. However, as remarked by Terence Tao, the uniqueness theory for hierarchy 1.5 is surprisingly delicate due to the fact that it is a system of infinitely many coupled equations over an unbounded number of variables. In [30], by imposing a space-time bound on the limit of $\left\{\gamma_N^{(k)}\right\}$, Klainerman and Machedon gave another proof of the uniqueness in [18] through a collapsing estimate originated from the ordinary multilinear Strichartz estimates in their null form paper [29] and a board game argument inspired by the Feynman graph argument in [18].

Later, the method in Klainerman and Machedon [30] was taken up by Kirkpatrick, Schlein, and Staffilani [28], who studied the corresponding problem in 2D; by Chen and Pavlović [5, 6], who considered the 1D and 2D 3-body interaction problem and the general existence theory of hierarchy 1.5; and by the author [12], who investigated the trapping problem in 2D. In [8, 9], Chen, Pavlović and Tzirakis worked out the virial and Morawetz identities for hierarchy 1.5.

Recently, for the 3D case without traps, Chen and Pavlović [7] proved that, for $\beta \in (0, 1/4)$, the limit of $\left\{\gamma_N^{(k)}\right\}$ actually satisfies the space-time bound assumed by Klainerman and Machedon [30] as $N \to \infty$. This has been a well-known open problem in the field. Moreover, they showed that the solution to the BBGKY hierarchy converges strongly to the solution to hierarchy 1.5 in H^1 without assuming asymptotically factorized initial datum. In this paper, we adapt and simplify their argument in establishing the Klainerman-Machedon space-time bound. We also extend the range of β from $(0, 1/4)$ in Chen and Pavlović [7] to $(0, 2/7]$. Through simple functional analysis, we obtain a convergence result without assuming asymptotically factorized initial datum as well. (See Corollary 1) But we are not claiming it as a main result in this paper. We compare our result and the one in Chen and Pavlović [7] briefly in Section 1.1 below.

1.1. **Comparison with Chen and Pavlović** [7]**.** For comparison purpose, we transform Theorem 4 in Section 2, which implies Theorem 1, into the general convergence result below without the assumption of asymptotically factorized initial data since the result in Chen and Pavlović [7] is under the same regularity setting (condition 1.6) as Theorem 4.

Corollary 1. *(For comparison purpose only, not a main result.) Let $\gamma_N^{(k)}(t, \mathbf{x}_k; \mathbf{x}'_k)$ be the k-marginal density associated with the solution of the N-body Schrödinger equation 1.2 for some $\beta \in (0, 2/7]$ and $\gamma^{(k)}(t, \mathbf{x}_k; \mathbf{x}'_k)$ be the solution to the Gross-Pitaevskii hierarchy with a quadratic trap* (*hierarchy* 3.3)*. Assume the initial data of $\gamma_N^{(k)}$ and $\gamma^{(k)}$ satisfy the following conditions*

(a′)
$$\lim_{N\to\infty} \operatorname{Tr}\left|\gamma_N^{(k)}(0, \mathbf{x}_k; \mathbf{x}'_k) - \gamma^{(k)}(0, \mathbf{x}_k; \mathbf{x}'_k)\right| = 0,$$

(b′)
$$\left\langle \psi_N(0), H_N^k \psi_N(0)\right\rangle \leqslant C^k N^k. \tag{1.6}$$

Then we have the convergence of the evolution in trace norm
$$\lim_{N\to\infty} \operatorname{Tr}\left|\gamma_N^{(k)}(t, \mathbf{x}_k; \mathbf{x}'_k) - \gamma^{(k)}(t, \mathbf{x}_k; \mathbf{x}'_k)\right| = 0.$$

Proof. This result for the non-trap case should be credited to Erdös, Schlein and Yau though they did not state it as one of their main theorems. They mentioned it on page 297 of their paper [21]. Once we have established Theorem 4, its proof together with some simple functional analysis proves this corollary. We include the proof in Appendix I (Section 9) for completeness. □

Briefly, the main result (Theorem 3.1) in Chen and Pavlović [7] is the following.

Theorem 2. [7] *Let $\omega = 0$. Suppose $\gamma_N^{(k)}(t, \mathbf{x}_k; \mathbf{x}'_k)$ is the k-marginal density associated with the solution of the N-body Schrödinger equation 1.2 for some $\beta \in (0, 1/4)$ and $\gamma^{(k)}(t, \mathbf{x}_k; \mathbf{x}'_k)$ is the solution to the ordinary Gross-Pitaevskii hierarchy without traps. If the initial data of $\gamma_N^{(k)}$ and $\gamma^{(k)}$ satisfy the following conditions*

(a′′)
$$\lim_{N\to\infty} \left\| \left(\prod_{j=1}^{k} \left(1 - \triangle_{x_j}\right)^{\frac{1}{2}+\varepsilon} \left(1 - \triangle_{x'_j}\right)^{\frac{1}{2}+\varepsilon}\right) \left(\gamma_N^{(k)}(0, \mathbf{x}_k; \mathbf{x}'_k) - \gamma^{(k)}(0, \mathbf{x}_k; \mathbf{x}'_k)\right)\right\|_{L^2(d\mathbf{x}_k d\mathbf{x}'_k)} = 0,$$

(b′′)
$$\left\langle \psi_N(0), H_N^k \psi_N(0)\right\rangle \leqslant C^k N^k,$$

then we have the convergence of the evolution in H^1 norm

$$\lim_{N\to\infty}\left\|\left(\prod_{j=1}^{k}\left(1-\triangle_{x_j}\right)^{\frac{1}{2}}\left(1-\triangle_{x_j'}\right)^{\frac{1}{2}}\right)\left(\gamma_N^{(k)}(t,\mathbf{x}_k;\mathbf{x}_k')-\gamma^{(k)}(t,\mathbf{x}_k;\mathbf{x}_k')\right)\right\|_{L^2(d\mathbf{x}_k d\mathbf{x}_k')}=0.$$

One can then easily tell the following: on the one hand, the result in this work allows a quadratic trap and a larger range of β in the analysis (these are also the main novelty and the main technical improvement of this paper); on the other hand, the result in [7] yields a stronger convergence (H^1 convergence) when the initial data admits a stronger convergence. The main purpose of this paper is to justify the mean-field approximation and offer a rigorous derivation of the 3D cubic NLS with a quadratic trap (Theorem 1). Thus we establish the convergence of probability densities which is the trace norm convergence in this context. The Chen-Pavlović result is crucial for their work on the Cauchy problem of Gross-Pitaevskii hierarchies. Before we delve into the proofs, we remark that taking the coupling level k to be $\ln N$ in Section 6.0.3 is exactly the place where we follow the original idea of Chen and Pavlović [7].

1.2. **The Anisotropic Version of the Main Theorem.** It is of interest to use anisotropic traps in laboratory experiments. (See, for example, [13].) Our proof for Theorem 1 also applies to the case with anisotropic traps. In fact, we have the following theorem.

Theorem 3. *Define the diagonal matrix*

$$Q=\begin{pmatrix}\omega_1^2 & 0 & 0\\ 0 & \omega_2^2 & 0\\ 0 & 0 & \omega_3^2\end{pmatrix}.$$

Let $\left\{\gamma_N^{(k)}\right\}$ be the family of marginal densities associated with ψ_N, the solution of the N-body Schrödinger equation

$$i\partial_t\psi_N=\sum_{j=1}^{N}\left(-\frac{1}{2}\triangle_{x_j}+\frac{1}{2}x_j^TQx_j\right)\psi_N+\frac{1}{N}\sum_{1\leqslant i<j\leqslant N}N^{3\beta}V(N^{\beta}(x_i-x_j))\psi_N$$

for some $\beta\in\left(0,\frac{2}{7}\right]$. Assume that the pair interaction V is a nonnegative $L^1(\mathbb{R}^3)\cap H^2(\mathbb{R}^3)\cap W^{2,\infty}(\mathbb{R}^3)$ spherically symmetric function. Moreover, suppose the initial datum of equation 1.2 verifies Conditions (a), (b), and (c) in the assumption of Theorem 1. Then $\forall t\geqslant 0$, $\forall k\geqslant 1$, we have the convergence in the trace norm that

$$\lim_{N\to\infty}\operatorname{Tr}\left|\gamma_N^{(k)}(t,\mathbf{x}_k;\mathbf{x}_k')-\prod_{j=1}^{k}\phi(t,x_j)\overline{\phi}(t,x_j')\right|=0,$$

where $\phi(t,x)$ is the solution to the 3D cubic NLS with anisotropic quadratic traps

$$\begin{aligned}i\partial_t\phi &= \left(-\frac{1}{2}\triangle_x+\frac{1}{2}x^TQx\right)\phi+b_0\left|\phi\right|^2\phi \text{ in } \mathbb{R}^{3+1}\\ \phi(0,x) &= \phi_0(x)\end{aligned}$$

and $b_0=\int_{\mathbb{R}^3}V(x)dx$.

The anisotropic version of the main theorem stated above yields to the same techniques as Theorem 1 but is technically more complicated. Therefore we prove only the latter in detail, merely suggesting during the course of the proof the appropriate modifications needed to obtain the more general theorem.

2. Proof of the Main Theorem (Theorem 1)

We establish Theorem 1 with a smooth approximation argument and the following theorem.

Theorem 4. *Let $\left\{\gamma_N^{(k)}\right\}$ be the family of marginal densities associated with ψ_N, the solution of the N-body Schrödinger equation 1.2 for some $\beta\in\left(0,\frac{2}{7}\right]$. Assume that the pair interaction V is a nonnegative $L^1(\mathbb{R}^3)\cap H^2(\mathbb{R}^3)\cap W^{2,\infty}(\mathbb{R}^3)$ spherically symmetric function. Moreover, suppose the initial datum of equation 1.2 is normalized, asymptotically factorized and verifies the condition that there is a C independent of k or N such that*

$$\left\langle\psi_N(0),H_N^k\psi_N(0)\right\rangle\leqslant C^kN^k$$

Then $\forall t\geqslant 0$ and $\forall k\geqslant 1$, we have the convergence in the trace norm that

$$\lim_{N\to\infty}\operatorname{Tr}\left|\gamma_N^{(k)}(t,\mathbf{x}_k;\mathbf{x}_k')-\prod_{j=1}^{k}\phi(t,x_j)\overline{\phi}(t,x_j')\right|=0,$$

where $\phi(t,x)$ *is the solution to the 3D cubic NLS with a quadratic trap*

$$\begin{aligned} i\partial_t\phi &= \left(-\frac{1}{2}\triangle_x+\omega^2\frac{|x|^2}{2}\right)\phi+b_0\left|\phi\right|^2\phi \text{ in } \mathbb{R}^{3+1} \\ \phi(0,x) &= \phi_0(x) \end{aligned}$$

and $b_0=\int_{\mathbb{R}^3}V(x)dx$.

Before presenting the proof of Theorem 4, we discuss how to deduce Theorem 1 from Theorem 4. It is a well-known smooth approximation argument. We include it for completeness. For technical details, we refer the readers to Erdös-Schlein-Yau [20, 21] and Kirkpatrick-Schlein-Staffilani [28].

Write the spaces of compact operators and trace class operators of $L^2(\mathbb{R}^{3k})$ as $\mathcal{K}_k$ and $\mathcal{L}_k^1$. Then $(\mathcal{K}_k)'=\mathcal{L}_k^1$. Via the fact that $\mathcal{K}_k$ is separable, we select a dense countable subset of the unit ball of $\mathcal{K}_k$ and call it $\left\{J_i^{(k)}\right\}_{i\geqslant 1}\subset\mathcal{K}_k$. We have $\left\|J_i^{(k)}\right\|\leqslant 1$ where $\|\cdot\|$ is the operator norm. We set up the following metric on $\mathcal{L}_k^1$, for $\gamma^{(k)}$, $\tilde{\gamma}^{(k)}\in\mathcal{L}_k^1$: define

$$d_k(\gamma^{(k)},\tilde{\gamma}^{(k)})=\sum_{i=1}^{\infty}2^{-i}\left|\operatorname{Tr}J_i^{(k)}\left(\gamma^{(k)}-\tilde{\gamma}^{(k)}\right)\right|.$$

Then a uniformly bounded sequence $\gamma_N^{(k)}\in\mathcal{L}_k^1$ converges to $\gamma^{(k)}\in\mathcal{L}_k^1$ with respect to the weak* topology if and only if

$$\lim_{N\to\infty}d_k(\gamma_N^{(k)},\gamma^{(k)})=0.$$

Fix $\kappa>0$ and $\chi\in C_c^\infty(\mathbb{R})$, with $0\leqslant\chi\leqslant 1$, $\chi(s)=1$, for $0\leqslant s\leqslant 1$, and $\chi(s)=0$, for $s\geqslant 2$. We regularize the initial data of the N-body Schrödinger equation 1.2 with

$$\tilde{\psi}_N(0)=\frac{\chi\left(\kappa H_N/N\right)\psi_N(0)}{\left\|\chi\left(\kappa H_N/N\right)\psi_N(0)\right\|},$$

and we denote $\tilde{\psi}_N(t)$ the solution of the N-body Schrödinger equation 1.2 subject to this regularized initial data and $\left\{\tilde{\gamma}_N^{(k)}(t)\right\}_{k=1}^{\infty}$ the family of marginal densities associated with $\tilde{\psi}_N(t)$.

With these notations, if $\kappa>0$ small enough, on the one hand, we have

$$\left\langle\tilde{\psi}_N(0),H_N^k\tilde{\psi}_N(0)\right\rangle\leqslant\tilde{C}^kN^k,$$

and

$$\lim_{N\to\infty}\operatorname{Tr}J^{(k)}\left(\tilde{\gamma}_N^{(k)}(0)-\prod_{j=1}^{k}\phi_0\overline{\phi}_0\right)=0,$$

for every $J^{(k)}\in\mathcal{K}_k$. Thus $\tilde{\gamma}_N^{(k)}(0,\mathbf{x}_k;\mathbf{x}_k')\to\prod_{j=1}^{k}\phi_0\overline{\phi}_0$ as $N\to\infty$ in the weak* topology. Since $\prod_{j=1}^{k}\phi_0\overline{\phi}_0$ is an orthogonal projection, the convergence in the weak* topology is equivalent to the convergence in the trace norm. Consequently, the conditions of Theorem 4 are verified and it implies that $\forall t\in[0,T_0]$ and $\forall k\geqslant 1$,

$$\lim_{N\to\infty}\operatorname{Tr}\left|\tilde{\gamma}_N^{(k)}(t,\mathbf{x}_k;\mathbf{x}_k')-\prod_{j=1}^{k}\phi(t,x_j)\overline{\phi}(t,x_j')\right|=0.$$

On the other hand, there is a constant C independent of N and κ such that

$$\left|\operatorname{Tr}J^{(k)}\left(\tilde{\gamma}_N^{(k)}(t)-\gamma_N^{(k)}(t)\right)\right|\leqslant C\left\|J^{(k)}\right\|\kappa^{\frac{1}{2}}$$

for every $J^{(k)}\in\mathcal{K}_k$. Therefore,

$$\begin{aligned} &\left|\operatorname{Tr}J^{(k)}\left(\gamma_N^{(k)}(t,\mathbf{x}_k;\mathbf{x}_k')-\prod_{j=1}^{k}\phi(t,x_j)\overline{\phi}(t,x_j')\right)\right| \\ \leqslant&\left|\operatorname{Tr}J^{(k)}\left(\gamma_N^{(k)}(t)-\tilde{\gamma}_N^{(k)}(t)\right)\right|+\left|\operatorname{Tr}J^{(k)}\left(\tilde{\gamma}_N^{(k)}(t)-\prod_{j=1}^{k}\phi(t,x_j)\overline{\phi}(t,x_j')\right)\right| \\ \leqslant&\ C\left\|J^{(k)}\right\|\kappa^{\frac{1}{2}}+\left|\operatorname{Tr}J^{(k)}\left(\tilde{\gamma}_N^{(k)}(t)-\prod_{j=1}^{k}\phi(t,x_j)\overline{\phi}(t,x_j')\right)\right|. \end{aligned}$$

As the above inequality holds for all $\kappa > 0$ small enough, we know $\gamma_N^{(k)}(t, \mathbf{x}_k; \mathbf{x}_k') \to \prod_{j=1}^k \phi(t, x_j)\overline{\phi}(t, x_j')$ as $N \to \infty$ in the weak* topology. This convergence is again equivalent to the convergence in the trace norm because $\prod_{j=1}^k \phi(t, x_j)\overline{\phi}(t, x_j')$ is an orthogonal projection as well. Whence we have established our main theorem (Theorem 1) through Theorem 4. It remains to show Theorem 4. We prove Theorem 4 with the help of the lens transform. We define the lens transform and show its related properties in the next section then we establish Theorem 4 in Section 4.

3. Lens Transform

In this section, we first define the lens transform and review its relevant properties, then we prove an energy estimate (Proposition 2) which relates the energy on the two sides of the lens transform. It aids in the proof of Theorem 4 in the sense that it links the analysis of $-\triangle_x + \omega^2 |x|^2$ to the analysis of $-\triangle_y$ which is a better understood operator. We denote (t, x) the space-time on the Hermite side and (τ, y) the space-time on the Laplacian side. We now define the lens transform we need.

Definition 1. *Let $\mathbf{x}_N, \mathbf{y}_N \in \mathbb{R}^{nN}$. We define the lens transform for L^2 functions $M_N : L^2(d\mathbf{y}_N) \to L^2(d\mathbf{x}_N)$ and its inverse by*

$$\begin{aligned}
(M_N u_N)(t, \mathbf{x}_N) &= \frac{e^{-i\omega \tan \omega t \frac{|\mathbf{x}_N|^2}{2}}}{(\cos \omega t)^{\frac{nN}{2}}} u_N\left(\frac{\tan \omega t}{\omega}, \frac{\mathbf{x}_N}{\cos \omega t}\right) \\
\left(M_N^{-1}\psi_N\right)(\tau, \mathbf{y}_N) &= \frac{e^{i\frac{\omega^2\tau}{1+\omega^2\tau^2}\frac{|\mathbf{y}_N|^2}{2}}}{(1+\omega^2\tau^2)^{\frac{nN}{4}}} \psi_N\left(\frac{\arctan(\omega\tau)}{\omega}, \frac{\mathbf{y}_N}{\sqrt{1+\omega^2\tau^2}}\right).
\end{aligned}$$

M_N is unitary by definition and the variables are related by

$$\tau = \frac{\tan \omega t}{\omega}, \ \mathbf{y}_N = \frac{\mathbf{x}_k}{\cos \omega t}.$$

Definition 2. *Let $\mathbf{x}_k, \mathbf{x}_k', \mathbf{y}_k, \mathbf{y}_k' \in \mathbb{R}^{nk}$. We define the lens transform for Hilbert-Schmidt kernels $T_k : L^2(d\mathbf{y}_k d\mathbf{y}_k') \to L^2(d\mathbf{x}_k d\mathbf{x}_k')$ and its inverse by*

$$\begin{aligned}
\left(T_k u^{(k)}\right)(t, \mathbf{x}_k; \mathbf{x}_k') &= \frac{e^{-i\omega \tan \omega t \frac{\left(|\mathbf{x}_k|^2 - |\mathbf{x}_k'|^2\right)}{2}}}{(\cos \omega t)^{nk}} u^{(k)}\left(\frac{\tan \omega t}{\omega}, \frac{\mathbf{x}_k}{\cos \omega t}; \frac{\mathbf{x}_k'}{\cos \omega t}\right) \\
\left(T_k^{-1}\gamma^{(k)}\right)(\tau, \mathbf{y}_k; \mathbf{y}_k') &= \frac{e^{i\frac{\omega^2\tau}{1+\omega^2\tau^2}\frac{\left(|\mathbf{y}_k|^2 - |\mathbf{y}_k'|^2\right)}{2}}}{(1+\omega^2\tau^2)^{\frac{nk}{2}}} \gamma^{(k)}\left(\frac{\arctan(\omega\tau)}{\omega}, \frac{\mathbf{y}_k}{\sqrt{1+\omega^2\tau^2}}; \frac{\mathbf{y}_k'}{\sqrt{1+\omega^2\tau^2}}\right).
\end{aligned}$$

T_k is unitary by definition as well and the variables are again related by

$$\tau = \frac{\tan \omega t}{\omega}, \ \mathbf{y}_k = \frac{\mathbf{x}_k}{\cos \omega t} \ \text{and} \ \mathbf{y}_k' = \frac{\mathbf{x}_k'}{\cos \omega t}.$$

Before we characterize the accurate effect of the lens transform, we clarify the motivation of such definitions by a lemma.

Lemma 1. [3, 12] *Define α and β via the system*

$$\begin{aligned}
\ddot{\alpha}(t) + \eta(t)\alpha(t) &= 0, \alpha(0) = 0, \dot{\alpha}(0) = 1, \\
\ddot{\beta}(t) + \eta(t)\beta(t) &= 0, \beta(0) = 1, \dot{\beta}(0) = 0.
\end{aligned}$$

If β is nonzero in the time interval $[0, T]$, then the solution of the 1D Schrödinger equation with a time dependent quadratic trap

$$\begin{aligned}
i\partial_t \psi &= \left(-\frac{1}{2}\partial_x^2 + \frac{1}{2}\eta(t)x^2\right)\psi \ \text{in} \ \mathbb{R}^{1+1} \\
\psi(0, x) &= f(x) \in L^2(\mathbb{R})
\end{aligned}$$

in $[0, T]$ is given by

$$\psi(t, x) = \frac{e^{i\frac{\dot{\beta}(t)}{\beta(t)}\frac{x^2}{2}}}{(\beta(t))^{\frac{1}{2}}} u\left(\frac{\alpha(t)}{\beta(t)}, \frac{x}{\beta(t)}\right),$$

if $u(\tau, y)$ solves the 1D free Schrödinger equation

$$i\partial_\tau u = -\frac{1}{2}\partial_y^2 u \ \text{in} \ \mathbb{R}^{1+1}$$

subject to the same initial data.

Proof. See [3] for a proof by direct computation and [12] for an algebraic proof using the metaplectic representation. When $\eta(t) = \omega^2$, such a transformation has a long history, we refer the readers to [3] and the references within. □

To make formulas shorter, let us write

$$\left(T_k^{-1}\gamma^{(k)}\right)(\tau, \mathbf{y}_k; \overrightarrow{\mathbf{y}'_k}) = \gamma^{(k)}(t, \mathbf{x}_k; \mathbf{x}'_k) h_n^{(k)}(\tau, \mathbf{y}_k; \mathbf{y}'_k)$$

then precisely, the lens transform has the following effect.

Proposition 1.

$$\begin{aligned}
&\left(i\partial_\tau + \frac{1}{2}\triangle_{\mathbf{y}_k} - \frac{1}{2}\triangle_{\mathbf{y}'_k}\right)\left(T_k^{-1}\gamma^{(k)}\right)(\tau, \mathbf{y}_k; \mathbf{y}'_k) \\
= &\frac{h_n^{(k)}(\tau, \mathbf{y}_k; \mathbf{y}'_k)}{(1+\omega^2\tau^2)}\left(i\partial_t - \left(-\frac{1}{2}\triangle_{\mathbf{x}_k} + \omega^2\frac{|\mathbf{x}_k|^2}{2}\right) + \left(-\frac{1}{2}\triangle_{\mathbf{x}'_k} + \omega^2\frac{|\mathbf{x}'_k|^2}{2}\right)\right)\left(\gamma^{(k)}\right)(t, \mathbf{x}_k; \mathbf{x}'_k)
\end{aligned}$$

Proof. This is a direct computation. □

Via Proposition 1, we know how the lens transform acts on the Bogoliubov–Born–Green–Kirkwood–Yvon (BBGKY) hierarchy and the Gross-Pitaevskii hierarchy.

Lemma 2. *(BBGKY hierarchy under the lens transform) Write* $V_N(x) = N^{3\beta}V\left(N^\beta x\right)$. $\left\{\gamma_N^{(k)}\right\}$ *solves the BBGKY hierarchy with a quadratic trap*

$$\begin{aligned}
&i\partial_t\gamma_N^{(k)} - \left(-\frac{1}{2}\triangle_{\mathbf{x}_k} + \omega^2\frac{|\mathbf{x}_k|^2}{2}\right)\gamma_N^{(k)} + \left(-\frac{1}{2}\triangle_{\mathbf{x}'_k} + \omega^2\frac{|\mathbf{x}'_k|^2}{2}\right)\gamma_N^{(k)} \qquad (3.1)\\
= &\frac{1}{N}\sum_{1\leqslant i<j\leqslant k}\left(V_N(x_i - x_j) - V_N(x'_i - x'_j)\right)\gamma_N^{(k)} \\
&+\frac{N-k}{N}\sum_{j=1}^{k}\int\left(V_N(x_j - x_{k+1}) - V_N(x'_j - x_{k+1})\right)\gamma_N^{(k+1)}(t, \mathbf{x}_k, x_{k+1}; \mathbf{x}'_k, x_{k+1})\,dx_{k+1},
\end{aligned}$$

in $[-T_0, T_0]$ *if and only if* $\left\{u_N^{(k)} = T_k^{-1}\gamma_N^{(k)}\right\}$ *solves the hierarchy*

$$\begin{aligned}
&\left(i\partial_\tau + \frac{1}{2}\triangle_{\mathbf{y}_k} - \frac{1}{2}\triangle_{\mathbf{y}'_k}\right)u_N^{(k)} \qquad (3.2)\\
= &\frac{1}{(1+\omega^2\tau^2)}\frac{1}{N}\sum_{1\leqslant i<j\leqslant k}\left(V_N\left(\frac{y_i - y_j}{(1+\omega^2\tau^2)^{\frac{1}{2}}}\right) - V_N\left(\frac{y'_i - y'_j}{(1+\omega^2\tau^2)^{\frac{1}{2}}}\right)\right)u_N^{(k)} \\
&+\frac{N-k}{N}\frac{1}{(1+\omega^2\tau^2)}\sum_{j=1}^{k}\int\left(V_N\left(\frac{y_j - y_{k+1}}{(1+\omega^2\tau^2)^{\frac{1}{2}}}\right) - V_N\left(\frac{y'_j - y_{k+1}}{(1+\omega^2\tau^2)^{\frac{1}{2}}}\right)\right) \\
&\times u_N^{(k+1)}(\tau, \mathbf{y}_k, y_{k+1}; \mathbf{y}'_k, y_{k+1})\,dy_{k+1},
\end{aligned}$$

in $\left[-\frac{\tan\omega T_0}{\omega}, \frac{\tan\omega T_0}{\omega}\right]$.

Lemma 3. *(Gross-Pitaevskii hierarchy under the lens transform)* $\left\{\gamma^{(k)}\right\}$ *solves the Gross-Pitaevskii hierarchy with a quadratic trap*

$$i\partial_t\gamma^{(k)} - \left(-\frac{1}{2}\triangle_{\mathbf{x}_k} + \omega^2\frac{|\mathbf{x}_k|^2}{2}\right)\gamma^{(k)} + \left(-\frac{1}{2}\triangle_{\mathbf{x}'_k} + \omega^2\frac{|\mathbf{x}'_k|^2}{2}\right)\gamma^{(k)} = b_0\sum_{j=1}^{k}B_{j,k+1}\gamma^{(k+1)}, \qquad (3.3)$$

in $[-T_0, T_0]$ *if and only if* $\left\{u^{(k)} = T_k^{-1}\gamma^{(k)}\right\}$ *solves the hierarchy*

$$\left(i\partial_\tau + \frac{1}{2}\triangle_{\mathbf{y}_k} - \frac{1}{2}\triangle_{\mathbf{y}'_k}\right)u^{(k)} = \frac{1}{(1+\omega^2\tau^2)^{\frac{2-n}{2}}}b_0\sum_{j=1}^{k}B_{j,k+1}u^{(k+1)}, \qquad (3.4)$$

in $\left[-\frac{\tan\omega T_0}{\omega}, \frac{\tan\omega T_0}{\omega}\right]$.

The lens transform for the Hilbert-Schmidt kernels is not only by definition a unitary transform on $L^2\left(\mathbb{R}^{2nk}\right)$, it is also an isometry on the space of the trace class operator kernels.

Lemma 4. *$\forall t\in[-T_0,T_0]$, $\forall K(\mathbf{y}_k,\mathbf{y}_k')$ the kernel of a trace class operator on $L^2\left(\mathbb{R}^{nk}\right)$. If*

$$\int K(\mathbf{y}_k,\mathbf{y}_k')f(\mathbf{y}_k')d\mathbf{y}_k'=\lambda f(\mathbf{y}_k),$$

then

$$\int\left(T_kK\right)\left(\mathbf{x}_k,\mathbf{x}_k'\right)\left(M_Nf\right)\left(\mathbf{x}_k'\right)d\mathbf{x}_k'=\lambda\left(M_Nf\right)\left(\mathbf{x}_k\right).$$

In other words, the eigenvectors of the kernel $\left(T_kK\right)\left(\mathbf{x}_k,\mathbf{x}_k'\right)$ are exactly the lens transform (Lemma 1) of the eigenvectors of the kernel $K(\mathbf{y}_k,\mathbf{y}_k')$ with the same eigenvalues. In particular, we have

$$\operatorname{Tr}\left|T_kK\right|=\operatorname{Tr}\left|K\right|.$$

Proof. This is a straight forward computation. We remark that we have defined the generalized lens transform for a function and a kernel separately via Definitions 1 and 2. □

Once we have proved the following proposition which relates the energy of the two sides of the lens transform, we can start the proof of Theorem 4.

Proposition 2. *Let $\psi_N(t,\mathbf{x}_N)$ be the solution to equation 1.2 for some $\beta\in(0,3/5)$ subject to initial $\psi_N(0)$ which satisfies the energy condition*

$$\left\langle\psi_N(0),H_N^k\psi_N(0)\right\rangle\leqslant C^kN^k.$$

If $u_N(\tau,\mathbf{y}_N)=M_N^{-1}\psi_N$, then there is a $C\geqslant0$, for all $k\geqslant0$, $\exists N_0\left(k\right)$ such that

$$\left\langle u_N(\tau),\prod_{j=1}^{k}\left(1-\triangle_{y_j}\right)u_N(\tau)\right\rangle\leqslant C^k,$$

for all $N\geqslant N_0$ and all $\tau\in\left[-\frac{\tan\omega T_0}{\omega},\frac{\tan\omega T_0}{\omega}\right]$ provided that $T_0<\frac{\pi}{2\omega}$.

The rest of this section is the proof of Proposition 2. We prove it for $\omega>0$ through Lemmas 5 and 6 since the case $\omega=0$ has already been studied in [15, 18].

Lemma 5. *For $\beta\in(0,3/5)$, there is a $C\geqslant0$, for all $k\geqslant0$, $\exists N_0\left(k\right)$ such that*

$$\left\langle\varphi,H_N^k\varphi\right\rangle\geqslant C^kN^k\left\langle\varphi,\prod_{j=1}^{k}\left(-\triangle_{x_j}+\omega^2\left|x_j\right|^2\right)\varphi\right\rangle,$$

for all $N\geqslant N_0$ and all $\varphi\in L_s^2\left(\mathbb{R}^{3N}\right)$.

Proof. The proof basically follows Proposition 1 in [15] step by step if one replaces $\left(1-\triangle_{x_j}\right)$ by $\left(-\triangle_{x_j}+\omega^2\left|x_j\right|^2\right)$ and notices that for $\omega>0$,

$$\left\|\left(1-\triangle\right)^{\frac{\alpha}{2}}f\right\|_{L^2}\leqslant C_\alpha\left\|\left(-\triangle+\omega^2\left|x\right|^2\right)^{\frac{\alpha}{2}}f\right\|_{L^2}\quad([38])$$

when one uses Sobolev. There are some extra error terms which can be easily handled. We illustrate the control of the extra error terms through the following example. Write $S_i^2=\left(-\triangle_{x_i}+\omega^2\left|x_i\right|^2\right)$, consider

$$\begin{aligned}&N^{3\beta}\left(\left\langle\varphi,S_1^2...S_{n+1}^2V\left(N^\beta\left(x_1-x_2\right)\right)\varphi\right\rangle+c.c.\right)\\=\;&N^{3\beta}\left(\left\langle S_3...S_{n+1}\varphi,S_1^2S_2^2V\left(N^\beta\left(x_1-x_2\right)\right)S_3...S_{n+1}\varphi\right\rangle+c.c.\right),\end{aligned}$$

where c.c. denotes complex conjugates. Neglecting $S_3...S_{n+1}$, we have

$$\begin{aligned}&N^{3\beta}\left(\left\langle\varphi,S_1^2S_2^2V\left(N^\beta\left(x_1-x_2\right)\right)\varphi\right\rangle+c.c.\right)\\=\;&N^{3\beta}\left(\left\langle\varphi,\left(-\triangle_{x_1}+\omega^2\left|x_1\right|^2\right)\left(-\triangle_{x_2}+\omega^2\left|x_2\right|^2\right)V\left(N^\beta\left(x_1-x_2\right)\right)\varphi\right\rangle+c.c.\right)\\=\;&N^{3\beta}\left(\left\langle\varphi,\left(-\triangle_{x_1}\right)\left(-\triangle_{x_2}\right)V\left(N^\beta\left(x_1-x_2\right)\right)\varphi\right\rangle+c.c.\right)\\&+N^{3\beta}\left(\left\langle\varphi,\left(-\triangle_{x_1}\right)\left(\omega^2\left|x_2\right|^2\right)V\left(N^\beta\left(x_1-x_2\right)\right)\varphi\right\rangle+c.c.\right)\\&+N^{3\beta}\left(\left\langle\varphi,\left(\omega^2\left|x_1\right|^2\right)\left(-\triangle_{x_2}\right)V\left(N^\beta\left(x_1-x_2\right)\right)\varphi\right\rangle+c.c.\right)\\&+N^{3\beta}\left(\left\langle\varphi,\left(\omega^2\left|x_1\right|^2\right)\left(\omega^2\left|x_2\right|^2\right)V\left(N^\beta\left(x_1-x_2\right)\right)\varphi\right\rangle+c.c.\right)\\=\;&I+II+III+IV\end{aligned}$$

Compared to [15], the extra error terms are II, III, and IV. Since IV is positive, we only look at

$$\begin{aligned} II &\geqslant -2N^{4\beta}\left|\left\langle \nabla_{x_1}\omega\left|x_2\right|\varphi,\left|(\nabla V)\left(N^{\beta}\left(x_1-x_2\right)\right)\right|\omega\left|x_2\right|\varphi\right\rangle\right| \\ &\geqslant -CN^{4\beta}\alpha\left\langle \nabla_{x_1}\omega\left|x_2\right|\varphi,\left|(\nabla V)\left(N^{\beta}\left(x_1-x_2\right)\right)\right|\nabla_{x_1}\omega\left|x_2\right|\varphi\right\rangle \\ &\quad -CN^{4\beta}\alpha^{-1}\left\langle \omega\left|x_2\right|\varphi,\left|(\nabla V)\left(N^{\beta}\left(x_1-x_2\right)\right)\right|\omega\left|x_2\right|\varphi\right\rangle \\ &\geqslant -CN^{4\beta}\left(\alpha\left\langle\varphi,S_1^2S_2^2\varphi\right\rangle+\alpha^{-1}N^{-2\beta}\left\langle\omega\left|x_2\right|\varphi,S_1^2\omega\left|x_2\right|\varphi\right\rangle\right) \text{ (Sobolev at the second term)} \\ &\geqslant -CN^{3\beta}\left\langle\varphi,S_1^2S_2^2\varphi\right\rangle \ \ (\alpha=N^{-\beta}) \\ &\geqslant -CN^{3\beta-2}N^2\left\langle\varphi,S_1^2S_2^2\varphi\right\rangle \end{aligned}$$

As long as $\beta\in\left(0,\frac{2}{3}\right)$, we can absorb the extra error terms into the main term $N^2\left\langle\varphi,S_1^2S_2^2\varphi\right\rangle$. The Sobolev we used is

$$\begin{aligned} \int V\left(N^{\beta}\left(x_1-x_2\right)\right)|\varphi|^2dx_1dx_2 &\leqslant \left\|V\left(N^{\beta}\cdot\right)\right\|_{L^{\frac{3}{2}}}\int\left(\int|\varphi|^6dx_1\right)^{\frac{1}{3}}dx_2 \\ &\leqslant CN^{-2\beta}\|V\|_{L^{\frac{3}{2}}}\int\left|\left(I-\triangle_{x_1}\right)^{\frac{1}{2}}\varphi\right|^2dx_1dx_2 \\ &\leqslant CN^{-2\beta}\|V\|_{L^{\frac{3}{2}}}\int\left|\left(-\triangle_{x_1}+\omega^2\left|x_1\right|^2\right)^{\frac{1}{2}}\varphi\right|^2dx_1dx_2. \end{aligned}$$

□

Lemma 6. *Let*

$$P_x(t)=i\nabla_x\cos\omega t-\omega x\sin\omega t.$$

If $u_N(\tau,\mathbf{y}_N)=M_N^{-1}(\psi_N)$, *then*

$$\begin{aligned} \left\langle u_N(\tau),\left(-\triangle_{y_j}\right)u_N(\tau)\right\rangle &= \left\langle P_{x_j}(t)\,\psi_N(t),P_{x_j}(t)\,\psi_N(t)\right\rangle \\ &= \left\langle\psi_N(t),P_{x_j}^2(t)\,\psi_N(t)\right\rangle. \end{aligned}$$

Proof. We provide a proof through direct computation here. We remark that $P_x(t)$ is in fact the evolution of momentum. See [12]. Without lose of generality, we may assume $N=1$, then

$$\begin{aligned} P_x(t)\,\psi_1(t) &= P_x(t)\left(\frac{e^{-i\frac{\omega\tan\omega t}{2}|x|^2}}{(\cos\omega t)^{\frac{3}{2}}}u_1\left(\frac{\tan\omega t}{\omega},\frac{x}{(\cos\omega t)}\right)\right) \\ &= \frac{e^{-i\frac{\omega\tan\omega t}{2}|x|^2}}{(\cos\omega t)^{\frac{3}{2}}}\left(i\nabla_x\cos\omega t\right)u_1\left(\frac{\tan\omega t}{\omega},\frac{x}{(\cos\omega t)}\right) \\ &= \frac{e^{-i\frac{\omega\tan\omega t}{2}|x|^2}}{(\cos\omega t)^{\frac{3}{2}}}\left(i\nabla_y\right)u_1(\tau,y). \end{aligned}$$

Thus

$$\left\langle u_1(\tau),\left(-\triangle_y\right)u_1(\tau)\right\rangle=\int\left|\nabla_yu_1(\tau,y)\right|^2dy=\int\frac{\left|\nabla_yu_1(\tau,y)\right|^2}{|\cos\omega t|^3}dx=\left\langle P_x(t)\,\psi_1(t),P_x(t)\,\psi_1(t)\right\rangle.$$

□

With the above lemmas, we prove Proposition 2. We first notice that

$$\left\langle u_N(\tau),\prod_{j=1}^k\left(1-\triangle_{y_j}\right)u_N(\tau)\right\rangle=\left\langle\psi_N(t),\prod_{j=1}^k\left(1+P_{x_j}^2(t)\right)\psi_N(t)\right\rangle \ \ (\text{Lemma 6}).$$

Since

$$\begin{aligned} &\left\langle f,\left(1+P_{x_j}^2(t)\right)f\right\rangle \\ =\ & \langle f,f\rangle+\cos^2\omega t\,\langle\nabla f,\nabla f\rangle+\omega^2\sin^2\omega t\,\langle xf,xf\rangle-2\omega\sin\omega t\cos\omega t\,\mathrm{Im}\,\langle\nabla f,xf\rangle \\ \leqslant\ & C\left\langle f,\left(-\triangle_x+\omega^2|x|^2\right)f\right\rangle, \end{aligned}$$

we know

$$\begin{aligned} \left\langle\psi_N(t),\prod_{j=1}^k\left(1+P_{x_j}^2(t)\right)\psi_N(t)\right\rangle &\leqslant C^k\left\langle\psi_N(t),\prod_{j=1}^k\left(-\triangle_{x_j}+\omega^2\left|x_j\right|^2\right)\psi_N(t)\right\rangle \\ &\leqslant \frac{C^k}{N^k}\left\langle\psi_N(t),H_N^k\psi_N(t)\right\rangle. \text{ (Lemma 5)} \end{aligned}$$

From the energy condition on the initial datum $\psi_N(0)$, we then deduce Proposition 2 that is

$$\left\langle u_N(\tau), \prod_{j=1}^{k}\left(1-\triangle_{y_j}\right)u_N(\tau)\right\rangle \leqslant C^k.$$

4. Proof of Theorem 4

We devote this section to establishing Theorem 4. The main idea is to first prove that, in the time period $\tau \in \left[0, \frac{\tan\omega T_0}{\omega}\right]$ with $T_0 < \frac{\pi}{2\omega}$, as $N \to \infty$, $\left\{u_N^{(k)} = T_k^{-1}\gamma_N^{(k)}\right\}$, the lens transform of the solution to the BBGKY hierarchy 3.1, converges to $\left\{u^{(k)} = T_k^{-1}\gamma^{(k)}\right\}$, the lens transform of the solution to the Gross-Pitaevskii hierarchy 3.3 in the trace norm, then use Lemma 4 to conclude the convergence $\gamma_N^{(k)} \to \gamma^{(k)}$ in the trace norm as $N \to \infty$ in the time period $t \in [0, T_0]$. Then the time translation invariance of equation 1.2 proves Theorem 4 for all time. We now present the proof in detail.

Step I. In Proposition 2, we have already established the energy estimate for $\left\{u_N = M_N^{-1}\psi_N\right\}$,

$$\sup_{\tau\in\left[0,\frac{\tan\omega T_0}{\omega}\right]}\left\langle u_N(\tau), \prod_{j=1}^{k}\left(1-\triangle_{y_j}\right)u_N(\tau)\right\rangle \leqslant C^k.$$

which becomes

$$\sup_{\tau\in\left[0,\frac{\tan\omega T_0}{\omega}\right]}\operatorname{Tr}\left(\prod_{j=1}^{k}\left(1-\triangle_{y_j}\right)\right)u_N^{(k)} \leqslant C^k, \tag{4.1}$$

for $\left\{u_N^{(k)} = T_k^{-1}\gamma_N^{(k)}\right\}$. Therefore we can utilize the proof in Erdös-Schlein-Yau [18] or Kirkpatrick-Schlein-Staffilani [28] to show that the sequence $\left\{u_N^{(k)} = T_k^{-1}\gamma_N^{(k)}\right\}$ is compact with respect to the weak* topology on the trace class operators and every limit point $\left\{u^{(k)}\right\}$ solves hierarchy 3.4 which is

$$\left(i\partial_\tau + \frac{1}{2}\triangle_{\mathbf{y}_k} - \frac{1}{2}\triangle_{\mathbf{y}_k'}\right)u^{(k)} = \sqrt{1+\omega^2\tau^2}\,b_0\sum_{j=1}^{k}B_{j,k+1}u^{(k+1)}, \tag{4.2}$$

in 3D. This is a fixed time argument. We omit the details here.

Notation 1. *To make formulas shorter, let us write*

$$g(\tau) = \left(1+\omega^2\tau^2\right)^{-\frac{1}{2}}$$

from here on. The only property of $g(\tau)$ we are going to need is that $0 < c \leqslant g(\tau) \leqslant C < \infty$ in any finite time period.

Step II. In this step, we use the a-priori estimate 4.1 to provide a space-time bound of $u^{(k)}$ so that we can employ Theorem 6 in Step III. We transform estimate 4.1 into the following theorem.

Theorem 5. *(Main Auxiliary Theorem) Assume V is a nonnegative $L^1(\mathbb{R}^3)\cap H^2(\mathbb{R}^3)$ function. Let*

$$\tilde{V}_{N,\tau}(y) = N^{3\beta}\tilde{V}_\tau(N^\beta y) = N^{3\beta}g^3(\tau)V(g(\tau)N^\beta y)$$

be the interaction potential with the interaction parameter $\beta \in \left(0, \frac{2}{7}\right]$. Suppose that $u_N^{(k)}$ solves hierarchy 3.2 in $[0,T] \subset \left[0, \frac{\tan\omega T_0}{\omega}\right]$, which, written in the integral form, is

$$\begin{aligned} u_N^{(k)} &= U^{(k)}(\tau)u_{N,0}^{(k)} \qquad (4.3)\\ &-\frac{i}{N}\sum_{1\leqslant i<j\leqslant k}\int_0^\tau U^{(k)}(\tau-s)\frac{\left(\tilde{V}_{N,s}(y_i-y_j)-\tilde{V}_{N,s}(y_i'-y_j')\right)}{g(s)}u_N^{(k)}(s,\mathbf{y}_k;\mathbf{y}_k')ds\\ &-i\frac{N-k}{N}\sum_{j=1}^{k}\int_0^\tau U^{(k)}(\tau-s)\left(\frac{\tilde{B}_{N,j,k+1,s}}{g(s)}u_N^{(k+1)}\right)ds, \end{aligned}$$

subject to the condition that

$$\sup_{\tau\in[0,T]}\operatorname{Tr}\left(\prod_{j=1}^{k}\left(1-\triangle_{y_j}\right)\right)u_N^{(k)} \leqslant C^k, \tag{4.4}$$

where $\tilde{B}_{N,j,k+1,\tau} = \tilde{B}^1_{N,j,k+1,\tau} - \tilde{B}^2_{N,j,k+1,\tau}$ *with*

$$\left(\tilde{B}^1_{N,j,k+1,\tau} u_N^{(k+1)}\right)(\tau, \mathbf{y}_k; \mathbf{y}'_k) = \int \tilde{V}_{N,\tau}(y_j - y_{k+1}) u_N^{(k+1)}(\tau, \mathbf{y}_k, y_{k+1}; \mathbf{y}'_k, y_{k+1}) dy_{k+1},$$
$$\left(\tilde{B}^2_{N,j,k+1,\tau} u_N^{(k+1)}\right)(\tau, \mathbf{y}_k; \mathbf{y}'_k) = \int \tilde{V}_{N,\tau}(y'_j - y_{k+1}) u_N^{(k+1)}(\tau, \mathbf{y}_k, y_{k+1}; \mathbf{y}'_k, y_{k+1}) dy_{k+1}.$$

and $U^{(k)}(\tau)$ *is the solution operator to the free equation, that is*

$$U^{(k)}(\tau) = e^{\frac{i\tau}{2}\triangle_{\mathbf{y}_k}} e^{-\frac{i\tau}{2}\triangle_{\mathbf{y}'_k}}.$$

Then there is a C *independent of* j, k *and* N *such that*

$$\int_0^T \left\| R^{(k)} \tilde{B}_{N,j,k+1,\tau} u_N^{(k+1)} \right\|_{L^2} d\tau \leqslant C^k.$$

where

$$R^{(k)} = \prod_{j=1}^k \left(|\nabla_{y_j}| \left|\nabla_{y'_j}\right| \right).$$

Proof. We prove our main auxiliary theorem in Section 6. This theorem establishes the Klainerman-Machedon space-time bound for $\beta \in \left(0, \frac{2}{7}\right]$. □

Via the above theorem, we infer that every limit point $\left\{u^{(k)}\right\}$ of $\left\{u_N^{(k)}\right\}$ satisfies the space time bound

$$\int_0^{\frac{\tan \omega T_0}{\omega}} \left\| R^{(k)} B_{j,k+1} u^{(k+1)}(\tau, \cdot; \cdot) \right\|_{L^2(\mathbb{R}^{3k} \times \mathbb{R}^{3k})} d\tau \leqslant C^k,$$

for some $C > 0$ and all $1 \leqslant j \leqslant k$.

Step III. Regarding the solution to the infinite hierarchy 4.2, we have the following uniqueness theorem.

Theorem 6. *Let* $\left\{u^{(k)}\right\}$ *be a solution of the infinite hierarchy 4.2 in* $[s,T] \subset \left[0, \frac{\tan \omega T_0}{\omega}\right]$ *subject to zero initial data that is*

$$u^{(k)}(s, \mathbf{y}_k; \mathbf{y}'_k) = 0, \forall k,$$

and the space time bound

$$\int_s^T \left\| R^{(k)} B_{j,k+1} u^{(k+1)}(\tau, \cdot; \cdot) \right\|_{L^2(\mathbb{R}^{3k} \times \mathbb{R}^{3k})} d\tau \leqslant C^k, \tag{4.5}$$

for some $C > 0$ *and all* $1 \leqslant j \leqslant k$. *Then* $\forall k, \tau \in [s,T]$, *we have*

$$\left\| R^{(k)} u^{(k)}(\tau, \cdot; \cdot) \right\|_{L^2(\mathbb{R}^{3k} \times \mathbb{R}^{3k})} = 0.$$

Proof. See Section 5. □

Since we have shown the space-time bound 4.5 in Step III, we apply the above uniqueness theorem and deduce that

$$u^{(k)}(\tau, \mathbf{y}_k; \mathbf{y}'_k) = \prod_{j=1}^k \tilde{\phi}(\tau, y_j) \overline{\tilde{\phi}}(\tau, y'_j), \tag{4.6}$$

where $\tilde{\phi}(\tau, y)$ solves the 3D NLS

$$\begin{aligned} i\partial_\tau \tilde{\phi} &= -\frac{1}{2}\triangle_y \tilde{\phi} + \frac{b_0 \left|\tilde{\phi}\right|^2 \tilde{\phi}}{g(\tau)} \text{ in } \mathbb{R}^{3+1} \\ \tilde{\phi}(0,y) &= \phi_0. \end{aligned} \tag{4.7}$$

Hence the compact sequence $\left\{u_N^{(k)}\right\}$ has only one limit point. So

$$u_N^{(k)} \to \prod_{j=1}^k \tilde{\phi}(\tau, y_j) \overline{\tilde{\phi}}(\tau, y'_j) \text{ as } N \to \infty$$

in the weak* topology. Since $u^{(k)}$ is an orthogonal projection, the convergence in the weak* topology is then equivalent to the convergence in the trace norm.

Example 1. *At the suggestion of Professor Walter Strauss, we give a brief explanation on why a factorized state like formula 4.6 is a solution to the Gross-Pitaevskii hierarchy. Consider $k=1$, then plugging $\tilde{\phi}(\tau,y_1)\overline{\tilde{\phi}}(\tau,y_1')$ into the infinite hierarchy yields*

$$\begin{aligned}
&\left(i\partial_\tau+\frac{1}{2}\triangle_{y_1}-\frac{1}{2}\triangle_{y_1'}\right)\left(\tilde{\phi}(\tau,y_1)\overline{\tilde{\phi}}(\tau,y_1')\right)\\
=\;& b_0\frac{\left(\left|\tilde{\phi}\right|^2\tilde{\phi}\right)(\tau,y_1)\overline{\tilde{\phi}}(\tau,y_1')}{g(\tau)}-b_0\frac{\tilde{\phi}(\tau,y_1)\left(\left|\tilde{\phi}\right|^2\overline{\tilde{\phi}}\right)(\tau,y_1')}{g(\tau)}\\
=\;& \frac{b_0B^1_{1,2}\left(\tilde{\phi}(\tau,y_1)\tilde{\phi}(\tau,y_2)\overline{\tilde{\phi}}(\tau,y_1')\overline{\tilde{\phi}}(\tau,y_2')\right)-b_0B^2_{1,2}\left(\tilde{\phi}(\tau,y_1)\tilde{\phi}(\tau,y_2)\overline{\tilde{\phi}}(\tau,y_1')\overline{\tilde{\phi}}(\tau,y_2')\right)}{g(\tau)},
\end{aligned}$$

which is

$$\left(i\partial_\tau+\frac{1}{2}\triangle_{y_1}-\frac{1}{2}\triangle_{y_1'}\right)u^{(1)}=\frac{1}{g(\tau)}b_0B_{1,2}u^{(2)}.$$

Step IV. In Step III, we have concluded the convergence

$$\lim_{N\to\infty}\operatorname{Tr}\left|u_N^{(k)}(\tau,\mathbf{y}_k;\mathbf{y}_k')-\prod_{j=1}^{k}\tilde{\phi}(\tau,y_j)\overline{\tilde{\phi}}(\tau,y_j')\right|=0,\ \forall\tau\in\left[0,\frac{\tan\omega T_0}{\omega}\right].$$

Notice that $u_N^{(k)}=T_k^{-1}\gamma_N^{(k)}$ and the lens transform of $u^{(k)}$ is

$$\gamma^{(k)}(t,\mathbf{x}_k;\mathbf{x}_k')=\prod_{j=1}^{k}\phi(t,x_j)\overline{\phi}(t,x_j'),$$

where $\phi(t,x)$ solves

$$\begin{aligned}
i\partial_t\phi &= \left(-\frac{1}{2}\triangle_x+\omega^2\frac{|x|^2}{2}\right)\phi+b_0\left|\phi\right|^2\phi\text{ in }\mathbb{R}^{3+1}\\
\phi(0,x) &= \phi_0(x)
\end{aligned}$$

i.e. equation 1.4. Thence we conclude that

$$\lim_{N\to\infty}\operatorname{Tr}\left|\gamma_N^{(k)}(t,\mathbf{x}_k;\mathbf{x}_k')-\prod_{j=1}^{k}\phi(t,x_j)\overline{\phi}(t,x_j')\right|=0,\ \forall t\in[0,T_0],$$

as a result of the fact that the lens transform preserves the trace norm (Lemma 4). Since equation 1.2 is time translation invariant, we have established Theorem 4 and consequently Theorem 1. The purpose of the rest of this paper is to prove Theorems 6 and 5.

Remark 1. *Since $\tilde{\phi}=M_1^{-1}\phi$, the global H^1 well-posedness of equation 4.7 is implied by the global well-posedness in the scattering space $\sum$ of equation 1.4 which comes from the Strichartz estimates. Through the lens transform, we always have a L^2 solution to equation 4.7. Lemma 6 then shows $\nabla\tilde{\phi}\in L^2$.*

5. The Uniqueness of Hierarchy 4.2 (Proof of Theorem 6)

In this section, we produce Theorem 6 with a collapsing estimate and the Klainerman-Machedon board game. We list these two tools in below. For convenience, we set the coupling constant b_0 in the infinite hierarchy 4.2 to be 1.

Lemma 7. [30] *Assume $u^{(k+1)}$ verifies*

$$\left(i\partial_\tau+\frac{1}{2}\triangle_{\mathbf{y}_{k+1}}-\frac{1}{2}\triangle_{\mathbf{y}_{k+1}'}\right)u^{(k+1)}=0,$$

then there is a $C>0$, independent of j, k, and $u^{(k+1)}$ s.t.

$$\begin{aligned}
&\left\|R^{(k)}\left(B_{j,k+1}u^{(k+1)}\right)(\tau,\mathbf{y}_k;\mathbf{y}_k')\right\|_{L^2(\mathbb{R}\times\mathbb{R}^{3k}\times\mathbb{R}^{3k})}\\
\leqslant\;& C\left\|R^{(k+1)}u^{(k+1)}(0,\mathbf{y}_{k+1};\mathbf{y}_{k+1}')\right\|_{L^2(\mathbb{R}^{3(k+1)}\times\mathbb{R}^{3(k+1)})}.
\end{aligned}$$

Proof. This is Theorem 1.3 of [30]. For some other collapsing estimates, see [10, 12, 22, 28]. ☐

Lemma 8. *Assuming zero initial data i.e.*

$$u^{(k)}(s,\mathbf{y}_k;\mathbf{y}_k')=0,\forall k,$$

then one can express $u^{(1)}(\tau_1,\cdot;\cdot)$ *in the Gross-Pitaevskii hierarchy 4.2 as a sum of at most* 4^n *terms of the form*

$$\int_D J(\underline{\tau}_{n+1},\mu_m)\left(u^{(n+1)}(\tau_{n+1})\right)d\underline{\tau}_{n+1},$$

or in other words,

$$u^{(1)}(\tau_1,\cdot;\cdot)=\sum_m\int_D J(\underline{\tau}_{n+1},\mu_m)\left(u^{(n+1)}(\tau_{n+1})\right)d\underline{\tau}_{n+1}. \tag{5.1}$$

Here $\underline{\tau}_{n+1}=(\tau_2,\tau_3,...,\tau_{n+1})$, $D\subset[s,\tau_1]^n$, μ_m *are a set of maps from* $\{2,...,n+1\}$ *to* $\{1,...,n\}$ *satisfying* $\mu_m(2)=1$ *and* $\mu_m(j)<j$ *for all* j, *and*

$$\begin{aligned}J(\underline{\tau}_{n+1},\mu_m)\left(u^{(n+1)}(\tau_{n+1})\right) &= \left(\prod_{j=1}^n g\left(\tau_{j+1}\right)\right)^{-1}U^{(1)}(\tau_1-\tau_2)B_{1,2}U^{(2)}(\tau_2-\tau_3)B_{\mu_m(3),2}\cdots\\ &\quad U^{(n)}(\tau_n-\tau_{n+1})B_{\mu_m(n+1),n+1}(u^{(n+1)}(\tau_{n+1},\cdot;\cdot)).\end{aligned}$$

Proof. The RHS of formula 5.1 is in fact an application of Duhamel's principle involving only the inhomogeneous terms since we have zero initial data. The parameter n is the coupling level we take. This lemma follows from the proof of Theorem 3.4 in [30]. One needs only notice that factors depending solely on τ, e.g.

$$\frac{1}{g\left(\tau_{j+1}\right)}$$

commutes with $U^{(k)}$ and $B_{j,k+1}$ $\forall j,k$. □

With Lemmas 7 and 8, we prove Theorem 6. Let $D_{\tau_2}=\{(\tau_3,...,\tau_{n+1})\,|\,(\tau_2,\tau_3,...,\tau_{n+1})\in D\}$ where D is as in Lemma 8. Given that we have already checked that

$$\left\|R^{(1)}u^{(1)}(s_0,\cdot)\right\|_{L^2(\mathbb{R}^3\times\mathbb{R}^3)}=0,$$

applying Lemma 8 to $[s_0,\tau_1]\subset[s,T]\subset[0,\frac{\tan\omega T_0}{\omega}]$, we have

$$\begin{aligned}
&\left\|R^{(1)}u^{(1)}(\tau_1,\cdot)\right\|_{L^2(\mathbb{R}^3\times\mathbb{R}^3)}\\
\leqslant\ &\sum_m\left\|R^{(1)}\int_D J(\underline{\tau}_{n+1},\mu_m)\left(u^{(n+1)}(\tau_{n+1})\right)d\underline{\tau}_{n+1}\right\|_{L^2(\mathbb{R}^3\times\mathbb{R}^3)}\\
\leqslant\ &\sum_m\int_{[s_0,\tau_1]^n}\left\|R^{(1)}J(\underline{\tau}_{n+1},\mu_m)\left(u^{(n+1)}(\tau_{n+1})\right)\right\|_{L^2(\mathbb{R}^3\times\mathbb{R}^3)}d\underline{\tau}_{n+1}\\
\leqslant\ &\sum_m\left(\inf_{\tau\in[0,\frac{\tan\omega T_0}{\omega}]}g(\tau)\right)^{-n}\int_{[s_0,\tau_1]^n}\left\|R^{(1)}U^{(1)}(\tau_1-\tau_2)B_{1,2}U^{(2)}(\tau_2-\tau_3)B_{\mu_m(3),2}\cdots\right\|_{L^2(\mathbb{R}^3\times\mathbb{R}^3)}d\underline{\tau}_{n+1}\\
=\ &\sum_m C^n\int_{[s_0,\tau_1]^n}\left\|R^{(1)}B_{1,2}U^{(2)}(\tau_2-\tau_3)B_{\mu_m(3),2}\cdots\right\|_{L^2(\mathbb{R}^3\times\mathbb{R}^3)}d\underline{\tau}_{n+1}\\
\leqslant\ &\sum_m C^n(\tau_1-s_0)^{\frac{1}{2}}\int_{[s_0,\tau_1]^{n-1}}\left(\int\left\|R^{(1)}B_{1,2}U^{(2)}(\tau_2-\tau_3)B_{\mu_m(3),2}\cdots\right\|^2_{L^2(\mathbb{R}^3\times\mathbb{R}^3)}d\tau_2\right)^{\frac{1}{2}}d\tau_3...d\tau_{n+1}\\
\leqslant\ &\sum_m C^nC(\tau_1-s_0)^{\frac{1}{2}}\int_{[s_0,\tau_1]^{n-1}}\left\|R^{(2)}B_{\mu_m(3),2}U^{(3)}(\tau_3-\tau_4)...\right\|_{L^2(\mathbb{R}^3\times\mathbb{R}^3)}d\tau_3...d\tau_{n+1}\ \text{(Lemma 7)}\\
&\text{(Iterate } n-2 \text{ times)}\\
&...\\
\leqslant\ &\sum_m C^n\left(C(\tau_1-s_0)\right)^{\frac{n-1}{2}}\int_{s_0}^{\tau_1}\left\|R^{(n)}B_{\mu_m(n+1),n+1}u^{(n+1)}(\tau_{n+1},\cdot;\cdot)\right\|_{L^2(\mathbb{R}^3\times\mathbb{R}^3)}d\tau_{n+1}\\
\leqslant\ &C\left(C(\tau_1-s_0)\right)^{\frac{n-1}{2}}.
\end{aligned}$$

Let (τ_1-s_0) be sufficiently small, and $n\to\infty$, we infer that

$$\left\|R^{(1)}u^{(1)}(\tau_1,\cdot)\right\|_{L^2(\mathbb{R}^3\times\mathbb{R}^3)}=0\text{ in }\left[s_0,\tau_1\right].$$

Such a choice of $(\tau_1 - s_0)$ works for all of $[s_0, T]$. Accordingly, we have $\left\|R^{(k)}u^{(k)}(\tau,\cdot)\right\|_{L^2(\mathbb{R}^3\times\mathbb{R}^3)} = 0$, $\forall k, \tau \in [s_0, T]$ by iterating the above argument. Thence we have attained Theorem 6.

6. The Space-Time Bound of the BBGKY Hierarchy for $\beta \in (0, 2/7]$ (Proof of Theorem 5)

We establish Theorem 5 in this section. This section also serves as a simplification and an extension of Chen-Pavlovic [7]. Without loss of generality, we may assume $k = 1$ that is

$$\int_0^T \left\|R^{(1)}\tilde{B}_{N,1,2,\tau}u_N^{(2)}\right\|_{L^2} d\tau \leqslant C. \tag{6.1}$$

We are going to prove estimate 6.1 for a sufficiently small time T determined by the controlling constant in condition 4.4 and independent of N, then the bootstrapping argument in Section 5 (Proof of Theorem 6) and condition 4.4 provide the bound for every finite time $T \in [0, \frac{\tan \omega T_0}{\omega}]$. Since we work with L^2 norms here, we transform condition 4.4 into the H^1 energy bound:

$$\int \left|\left(\prod_{j=1}^{k}\left(1-\triangle_{y_j}\right)^{\frac{1}{2}}\left(1-\triangle_{y_j'}\right)^{\frac{1}{2}}\right)u_N^{(k)}\right|^2 d\mathbf{y}_k d\mathbf{y}_k' \leqslant \left(\operatorname{Tr}\left(\prod_{j=1}^{k}\left(1-\triangle_{y_j}\right)\right)u_N^{(k)}\right)^2.$$

To obtain the above estimate, one notices

$$\begin{aligned}
&\int\left|\left(1-\triangle_y\right)^{\frac{1}{2}}\left(1-\triangle_{y'}\right)^{\frac{1}{2}}\int\phi\left(y,r\right)\overline{\phi\left(y',r\right)}dr\right|^2 dydy' \\
=\ &\int\left|\int\left(1-\triangle_y\right)^{\frac{1}{2}}\phi\left(y,r\right)\overline{\left(1-\triangle_{y'}\right)^{\frac{1}{2}}\phi\left(y',r\right)}dr\right|^2 dydy' \\
\leqslant\ &\int\left(\int\left(1-\triangle_y\right)^{\frac{1}{2}}\phi\left(y,r\right)\overline{\left(1-\triangle_y\right)^{\frac{1}{2}}\phi\left(y,r\right)}dr\right)\left(\int\left(1-\triangle_{y'}\right)^{\frac{1}{2}}\phi\left(y',r\right)\overline{\left(1-\triangle_{y'}\right)^{\frac{1}{2}}\phi\left(y',r\right)}dr\right)dydy' \\
=\ &\left(\int\phi\left(y,r\right)\overline{\left(1-\triangle_y\right)\phi\left(y,r\right)}dydr\right)^2,
\end{aligned}$$

the energy bound then follows from the definition of $u_N^{(k)}$. The analysis of Theorem 5 also involves $\tilde{B}_{N,j,k+1,\tau}$ which approximates $B_{j,k+1}$ for every τ, so we generalize Lemma 7 to the following collapsing estimate.

Theorem 7. *Suppose $u(\tau, y_1, y_2, y_2')$ solves the Schrödinger equation*

$$\begin{aligned}
iu_\tau + \frac{1}{2}\triangle_{y_1}u + \frac{1}{2}\triangle_{y_2}u - \frac{1}{2}\triangle_{y_2'}u &= 0 \text{ in } \mathbb{R}^{9+1} \\
u(0, y_1, y_2, y_2') &= f(y_1, y_2, y_2'),
\end{aligned}$$

and $g(\tau) \geqslant c_0 > 0$, then there is a C independent of N and u such that

$$\begin{aligned}
&\int_{\mathbb{R}^{3+1}}\left|\left|\nabla_y\right|\left(\int\left(g^3(\tau)V_N(g(\tau)\left(y-y_2\right))\right)\delta(y_2-y_2')u(\tau,y,y_2;y_2')dy_2dy_2'\right)\right|^2 dyd\tau \\
\leqslant\ & Cb_0^2\left\|\left|\nabla_{y_1}\right|\left|\nabla_{y_2}\right|\left|\nabla_{y_2'}\right|f\right\|_2^2.
\end{aligned}$$

where $b_0 = \int V_N dy$.

Proof. Theorem 7 follows from a slightly modified version of the proof of Theorem 2 of [12]. We include it in Appendix II for completeness. □

We now present the proof of estimate 6.1. In order to more conveniently apply the Klainerman-Machedon board game, let us start by rewriting hierarchy 4.3 as

$$\begin{aligned}
u_N^{(k)}(\tau_k) &= U^{(k)}(\tau_k)u_{N,0}^{(k)} + \int_0^{\tau_k}U^{(k)}(\tau_k-\tau_{k+1})\frac{\tilde{V}_{N,\tau_{k+1}}^{(k)}u_N^{(k)}(\tau_{k+1})}{g\left(\tau_{k+1}\right)}d\tau_{k+1} \\
&\quad + \frac{N-k}{N}\int_0^{\tau_k}U^{(k)}(\tau_k-\tau_{k+1})\frac{\tilde{B}_{N,\tau_{k+1}}^{(k+1)}u_N^{(k+1)}(\tau_{k+1})}{g\left(\tau_{k+1}\right)}d\tau_{k+1}
\end{aligned} \tag{6.2}$$

where

$$\begin{aligned}
\tilde{V}_{N,\tau}^{(k)}u_N^{(k)}(\tau,\mathbf{y}_k;\mathbf{y}_k') &= \frac{1}{N}\sum_{1\leqslant i<j\leqslant k}\left(\tilde{V}_{N,\tau}(y_i-y_j)-\tilde{V}_{N,\tau}(y_i'-y_j')\right)u_N^{(k)}(\tau,\mathbf{y}_k;\mathbf{y}_k') \\
\tilde{B}_{N,\tau}^{(k+1)}u_N^{(k+1)} &= \sum_{j=1}^{k}\tilde{B}_{N,j,k+1,\tau}u_N^{(k+1)} = \sum_{j=1}^{k}\left(\tilde{B}_{N,j,k+1,\tau}^1-\tilde{B}_{N,j,k+1,\tau}^2\right)u_N^{(k+1)}.
\end{aligned}$$

We omit the i in front of the potential term and the interaction term so that we do not need to keep track of its exact power.

Iterate Duhamel's principle (equation 6.2), we have

$$\begin{aligned}
& u_N^{(2)}(\tau_2) \\
= & U^{(2)}(\tau_2)u_{N,0}^{(2)} + \int_0^{\tau_2} U^{(2)}(\tau_2-\tau_3)\frac{\tilde{V}_{N,\tau_3}^{(2)}u_N^{(2)}(\tau_3)}{g(\tau_3)}d\tau_3 + \frac{N-2}{N}\int_0^{\tau_2} U^{(2)}(\tau_2-\tau_3)\frac{\tilde{B}_{N,\tau_3}^{(3)}u_N^{(3)}(\tau_3)}{g(\tau_3)}d\tau_3 \\
= & U^{(2)}(\tau_2)u_{N,0}^{(2)} + \frac{N-2}{N}\int_0^{\tau_2} U^{(2)}(\tau_2-\tau_3)\frac{\tilde{B}_{N,\tau_3}^{(3)}U^{(3)}(\tau_3)u_{N,0}^{(3)}}{g(\tau_3)}d\tau_3 + \int_0^{\tau_2} U^{(2)}(\tau_2-\tau_3)\frac{\tilde{V}_{N,\tau_3}^{(2)}u_N^{(2)}(\tau_3)}{g(\tau_3)}d\tau_3 \\
& + \frac{N-2}{N}\int_0^{\tau_2} U^{(2)}(\tau_2-\tau_3)\frac{\tilde{B}_{N,\tau_3}^{(3)}}{g(\tau_3)}\int_0^{\tau_3} U^{(3)}(\tau_3-\tau_4)\frac{\tilde{V}_{N,\tau_4}^{(3)}u_N^{(3)}(\tau_4)}{g(\tau_4)}d\tau_4 d\tau_3 \\
& + \frac{N-2}{N}\frac{N-3}{N}\int_0^{\tau_2} U^{(2)}(\tau_2-\tau_3)\frac{\tilde{B}_{N,\tau_3}^{(3)}}{g(\tau_3)}\int_0^{\tau_3} U^{(3)}(\tau_3-\tau_4)\frac{\tilde{B}_{N,\tau_4}^{(4)}u_N^{(4)}(\tau_4)}{g(\tau_4)}d\tau_4 d\tau_3 \\
& ... \\
= & FreePart^{(k)} + PotentialPart^{(k)} + InteractionPart^{(k)}
\end{aligned}$$

where

$$\begin{aligned}
& FreePart^{(k)} \\
= & U^{(2)}(\tau_2)u_{N,0}^{(2)} + \sum_{j=3}^{k}\left(\prod_{l=3}^{j}\frac{N+1-l}{N}\right) \\
& \times \int_0^{\tau_2}...\int_0^{\tau_{j-1}}\left(\prod_{l=2}^{j-1}g(\tau_{l+1})\right)^{-1}U^{(2)}(\tau_2-\tau_3)\tilde{B}_{N,\tau_3}^{(3)}...U^{(j-1)}(\tau_{j-1}-\tau_j)\tilde{B}_{N,\tau_j}^{(j)} \\
& \times\left(U^{(j)}(\tau_j)u_{N,0}^{(j)}\right)d\tau_3...d\tau_j,
\end{aligned}$$

$$\begin{aligned}
& PotentialPart^{(k)} \\
= & \int_0^{\tau_2} U^{(2)}(\tau_2-\tau_3)\frac{\tilde{V}_{N,\tau_3}^{(2)}u_N^{(2)}(\tau_3)}{g(\tau_3)}d\tau_3 + \sum_{j=3}^{k}\left(\prod_{l=3}^{j}\frac{N+1-l}{N}\right) \\
& \times \int_0^{\tau_2}...\int_0^{\tau_{j-1}}\left(\prod_{l=2}^{j-1}g(\tau_{l+1})\right)^{-1}U^{(2)}(\tau_2-\tau_3)\tilde{B}_{N,\tau_3}^{(3)}...U^{(j-1)}(\tau_{j-1}-\tau_j)\tilde{B}_{N,\tau_j}^{(j)} \\
& \times\left(\int_0^{\tau_j} U^{(j)}(\tau_j-\tau_{j+1})\frac{\tilde{V}_{N,\tau_{j+1}}^{(j)}u_N^{(j)}(\tau_{j+1})}{g(\tau_{j+1})}d\tau_{j+1}\right)d\tau_3...d\tau_j,
\end{aligned}$$

$$\begin{aligned}
& InteractionPart^{(k)} \\
= & \left(\prod_{l=3}^{k+1}\frac{N+1-l}{N}\right) \\
& \times \int_0^{\tau_2}...\int_0^{\tau_k}\left(\prod_{l=2}^{k}g(\tau_{l+1})\right)^{-1}U^{(2)}(\tau_2-\tau_3)\tilde{B}_{N,\tau_3}^{(3)}...U^{(k)}(\tau_k-\tau_{k+1})\tilde{B}_{N,\tau_{k+1}}^{(k+1)} \\
& \times\left(u_N^{(k+1)}(\tau_{k+1})\right)d\tau_3...d\tau_{k+1}
\end{aligned}$$

From here on out, the k's in the formulas are the coupling level we take to prove estimate 6.1, it is distinct from the k in the statement of Theorem 5.

We are going to argue

$$\int_0^T \left\|R^{(1)}\tilde{B}_{N,1,2,\tau_2}FreePart^{(k)}(\tau_2)\right\|_{L^2}d\tau_2 \leqslant C \tag{6.3}$$

$$\int_0^T \left\|R^{(1)}\tilde{B}_{N,1,2,\tau_2}PotentialPart^{(k)}(\tau_2)\right\|_{L^2}d\tau_2 \leqslant C \tag{6.4}$$

$$\int_0^T \left\|R^{(1)}\tilde{B}_{N,1,2,\tau_2}InteractionPart^{(k)}(\tau_2)\right\|_{L^2}d\tau_2 \leqslant C \tag{6.5}$$

for some C and a sufficiently small T determined by the controlling constant in condition 4.4 and independent of N. We observe that $\tilde{B}_{N,\tau_j}^{(j)}$ has $2j$ terms inside so that each summand of $u_N^{(2)}(\tau_2)$ contains factorially many terms ($\sim k!$). So we use the Klainerman-Machedon board game to reduce the number of terms. Define

$$J_N(\underline{\tau}_{j+1})(f) = \left(\prod_{l=2}^{j} g(\tau_{l+1})\right)^{-1} U^{(2)}(\tau_2-\tau_3)\tilde{B}_{N,\tau_3}^{(3)}...U^{(j)}(\tau_j-\tau_{j+1})\tilde{B}_{N,\tau_{j+1}}^{(j+1)}f,$$

where $\underline{\tau}_{j+1}$ means $(\tau_3,...,\tau_{j+1})$, then the Klainerman-Machedon board game implies the lemma.

Lemma 9. [30] *One can express*

$$\int_0^{\tau_2}...\int_0^{\tau_j} J_N(\underline{\tau}_{j+1})(f)d\underline{\tau}_{j+1}$$

as a sum of at most 4^{j-1} *terms of the form*

$$\int_D J_N(\underline{\tau}_{j+1},\mu_m)(f)d\underline{\tau}_{j+1},$$

or in other words,

$$\int_0^{\tau_2}...\int_0^{\tau_j} J_N(\underline{\tau}_{j+1})(f)d\underline{\tau}_{j+1} = \sum_m \int_D J_N(\underline{\tau}_{j+1},\mu_m)(f)d\underline{\tau}_{j+1}.$$

Here $D\subset[0,\tau_2]^{j-1}$, μ_m *are a set of maps from* $\{3,...,j+1\}$ *to* $\{2,...,j\}$ *satisfying* $\mu_m(3)=2$ *and* $\mu_m(l)<l$ *for all* l, *and*

$$\begin{aligned} J_N(\underline{\tau}_{j+1},\mu_m)(f) &= \left(\prod_{l=2}^{j} g(\tau_{l+1})\right)^{-1} U^{(2)}(\tau_2-\tau_3)\tilde{B}_{N,2,3,\tau_3}U^{(3)}(\tau_3-\tau_4)\tilde{B}_{N,\mu_m(4),4,\tau_4}\cdots \\ &\quad U^{(j)}(\tau_j-\tau_{j+1})\tilde{B}_{N,\mu_m(j+1),j+1,\tau_{j+1}}(f). \end{aligned}$$

Remark 2. *There is no difference between Lemma 9 and the one we used for the uniqueness of hierarchy 4.2 (Lemma 8). We have restated it to remind the reader of its exact form since we start from* $u_N^{(2)}$ *here.*

With the above lemma and the collapsing estimate (Theorem 7), we have the following relation, which is essentially part of the proof of Theorem 6, to help establishing estimates 6.3, 6.4, and 6.5.

$$\begin{aligned} &\int_0^T \left\| R^{(1)}\tilde{B}_{N,1,2,\tau_2}\int_D J_N(\underline{\tau}_{j+1},\mu_m)(f)d\underline{\tau}_{j+1}\right\|_{L^2} d\tau_2 \qquad (6.6)\\ =\ &\int_0^T \left\| \int_D \left(\prod_{l=2}^{j} g(\tau_{l+1})\right)^{-1} R^{(1)}\tilde{B}_{N,1,2,\tau_2}U^{(2)}(\tau_2-\tau_3)\tilde{B}_{N,2,3,\tau_3}...d\tau_3...d\tau_{j+1}\right\|_{L^2} d\tau_2 \\ \leqslant\ &\left(\inf_{\tau\in[0,\frac{\tan\omega T_0}{\omega}]} g(\tau)\right)^{-j+1} \int_{[0,T]^j} \left\| R^{(1)}\tilde{B}_{N,1,2,\tau_2}U^{(2)}(\tau_2-\tau_3)\tilde{B}_{N,2,3,\tau_3}...\right\|_{L^2} d\tau_2 d\tau_3...d\tau_{j+1} \\ \leqslant\ &C^{j-1}T^{\frac12}\int_{[0,T]^{j-1}}\left(\int \left\| R^{(1)}\tilde{B}_{N,1,2,\tau_2}U^{(2)}(\tau_2-\tau_3)\tilde{B}_{N,2,3,\tau_3}...\right\|_{L^2}^2 d\tau_2\right)^{\frac12} d\tau_3...d\tau_{j+1} \\ &\text{(Cauchy-Schwarz)} \\ \leqslant\ &C^{j-1}\left(CT^{\frac12}\right)\int_{[0,T]^{j-1}}\left\| R^{(2)}\tilde{B}_{N,2,3,\tau_3}U^{(3)}(\tau_3-\tau_4)...\right\| d\tau_3...d\tau_{j+1}\ \ (\textit{Theorem } 7) \\ &\text{(Iterate } j-2 \text{ times)} \\ &... \\ \leqslant\ &(CT^{\frac12})^{j-1}\int_0^T \left\| R^{(j)}\tilde{B}_{N,\mu_m(j+1),j+1,\tau_{j+1}}f\right\|_{L^2} d\tau_{j+1}. \end{aligned}$$

We show estimate 6.3 in Section 6.0.1. Assuming Proposition 3, whose proof is postponed to Section 6.0.4, we derive estimate 6.4 in Section 6.0.2. Finally, by taking the coupling level k to be $\ln N$, we check estimate 6.5 in Section 6.0.3. We remark that the proof of estimates 6.3 and 6.4 is independent of the choice of the coupling level k. Taking the coupling level k to be $\ln N$ in the estimate of the interaction part is exactly the place where we follow the original idea of Chen and Pavlović [7]. Moreover, the proof of estimate 6.4 (Section 6.0.2) is the only place which relies on $\beta\in\left(0,\frac27\right]$ in this paper.

6.0.1. *Estimate of the Free Part of $u_N^{(2)}$.* Applying Lemma 9 and relation 6.6 to the free part of $u_N^{(2)}$, we obtain

$$
\begin{aligned}
&\int_0^T \left\| R^{(1)} \tilde{B}_{N,1,2,\tau_2} FreePart^{(k)}(\tau_2) \right\|_{L^2} d\tau_2 \\
\leqslant\ & CT^{\frac{1}{2}} \left\| R^{(2)} u_{N,0}^{(2)} \right\|_{L^2} + \sum_{j=3}^{k} \sum_m \int_0^T \left\| R^{(1)} \tilde{B}_{N,1,2,\tau_2} \int_D J_N(\underline{\tau}_j, \mu_m)(U^{(j)}(\tau_j) u_{N,0}^{(j)}) d\underline{\tau}_j \right\|_{L^2} d\tau_2 \\
\leqslant\ & CT^{\frac{1}{2}} \left\| R^{(2)} u_{N,0}^{(2)} \right\|_{L^2} + \sum_{j=3}^{k} \sum_m C(CT^{\frac{1}{2}})^{j-2} \int_0^T \left\| R^{(j-1)} \tilde{B}_{N,\mu_m(j),j,\tau_j} U^{(j)}(\tau_j) u_{N,0}^{(j)} \right\|_{L^2} d\tau_j. \\
\leqslant\ & CT^{\frac{1}{2}} \left\| R^{(2)} u_{N,0}^{(2)} \right\|_{L^2} + C \sum_{j=3}^{k} 4^{j-2} (CT^{\frac{1}{2}})^{j-1} \left\| R^{(j)} u_{N,0}^{(j)} \right\|_{L^2} \\
\leqslant\ & CT^{\frac{1}{2}} \left\| R^{(2)} u_{N,0}^{(2)} \right\|_{L^2} + C \sum_{j=3}^{\infty} (CT^{\frac{1}{2}})^{j-1} C^j \text{ (Condition 4.4)} \\
\leqslant\ & C < \infty \text{ for } T \text{ small enough.}
\end{aligned}
$$

Whence, we have shown estimate 6.3.

6.0.2. *Estimate of the Potential Part of $u_N^{(2)}$.* The same procedure in Section 6.0.1 deduces

$$
\begin{aligned}
&\int_0^T \left\| R^{(1)} \tilde{B}_{N,1,2,\tau_2} PotentialPart^{(k)}(\tau_2) \right\|_{L^2} d\tau_2 \\
\leqslant\ & \int_0^T \left\| \int_0^{\tau_2} R^{(1)} \tilde{B}_{N,1,2,\tau_2} U^{(2)}(\tau_2 - \tau_3) \frac{\tilde{V}_{N,\tau_3}^{(2)} u_N^{(2)}(\tau_3)}{g(\tau_3)} d\tau_3 \right\|_{L^2} d\tau_2 \\
& + \sum_{j=3}^{k} \sum_m \int_0^T \left\| R^{(1)} \tilde{B}_{N,1,2,\tau_2} \int_D J_N(\underline{\tau}_j, \mu_m) (\int_0^{\tau_j} U^{(j)}(t_j - t_{j+1}) \frac{\tilde{V}_{N,\tau_{j+1}}^{(j)} u_N^{(j)}(\tau_{j+1})}{g(\tau_{j+1})} d\tau_{j+1}) d\underline{\tau}_j \right\|_{L^2} d\tau_2 \\
\leqslant\ & CT^{\frac{1}{2}} \int_0^T \left\| R^{(2)} \tilde{V}_{N,\tau_3}^{(2)} u_N^{(2)}(\tau_3) \right\|_{L^2} d\tau_3 \\
& + \sum_{j=3}^{k} \sum_m C(CT^{\frac{1}{2}})^{j-2} \int_0^T \left\| R^{(j-1)} \tilde{B}_{N,\mu_m(j),j,\tau_j} \int_0^{\tau_j} U^{(j)}(\tau_j - \tau_{j+1}) \frac{\tilde{V}_{N,\tau_{j+1}}^{(j)} u_N^{(j)}(\tau_{j+1})}{g(\tau_{j+1})} d\tau_{j+1} \right\|_{L^2} d\tau_j \\
\leqslant\ & CT^{\frac{1}{2}} \int_0^T \left\| R^{(2)} \tilde{V}_{N,\tau_3}^{(2)} u_N^{(2)}(\tau_3) \right\|_{L^2} d\tau_3 + C \sum_{j=3}^{k} 4^{j-2} (CT^{\frac{1}{2}})^{j-1} \left(\int_0^T \left\| R^{(j)} \tilde{V}_{N,\tau_{j+1}}^{(j)} u_N^{(j)}(\tau_{j+1}) \right\|_{L^2} d\tau_{j+1} \right).
\end{aligned}
$$

Assume for the moment that we have the estimate

$$\int_0^T \left\| R^{(k)} \tilde{V}_{N,\tau}^{(k)} u_N^{(k)}(\tau) \right\|_{L^2} d\tau \leqslant C_0 C^k T$$

where C and C_0 are independent of T, k and N, then

$$\int_0^T \left\| R^{(1)} \tilde{B}_{N,1,2,\tau_2} PotentialPart^{(k)}(\tau_2) \right\|_{L^2} d\tau_2 \leqslant C < \infty.$$

for a sufficiently small T and a C independent of k and N. As a result, we complete the proof of estimate 6.4 with the following proposition.

Proposition 3. *Assume $\beta \in \left(0, \frac{2}{7}\right]$ and V is a nonnegative $L^1(\mathbb{R}^3) \cap H^2(\mathbb{R}^3)$ function. Then, given $T \in \left[0, \frac{\tan \omega T_0}{\omega}\right]$, there are C and C_0 independent of T, k and N such that*

$$\int_0^T \left\| R^{(k)} \tilde{V}_{N,\tau}^{(k)} u_N^{(k)}(\tau) \right\|_{L^2} d\tau \leqslant C_0 C^k T.$$

Proof. The proof is elementary and we relegate it to Section 6.0.4. □

Remark 3. *This proposition is exactly the reason we restrict $\beta \in \left(0, \frac{2}{7}\right]$ in this paper.*

6.0.3. *Estimate of the Interaction Part of* $u_N^{(2)}$. We proceed like Sections 6.0.1 and 6.0.2.

$$\begin{aligned}
&\int_0^T \left\| R^{(1)}\tilde{B}_{N,1,2,\tau_2} InteractionPart^{(k)}(\tau_2)\right\|_{L^2} d\tau_2 \\
\leqslant\ & \sum_m \int_0^T \left\| R^{(1)}\tilde{B}_{N,1,2,\tau_2} \int_D J_N(\underline{\tau}_{k+1},\mu_m)(u_N^{(k+1)}(\tau_{k+1}))d\underline{\tau}_{k+1}\right\|_{L^2} d\tau_2 \\
\leqslant\ & \sum_m C(CT^{\frac{1}{2}})^{k-1} \int_0^T \left\| R^{(k)}\tilde{B}_{N,\mu_m(k+1),k+1,\tau_{k+1}} u_N^{(k+1)}(\tau_{k+1})\right\|_{L^2} d\tau_{k+1}.
\end{aligned}$$

Then the next step is to investigate

$$\int_0^T \left\| R^{(k)}\tilde{B}_{N,\mu_m(k+1),k+1,\tau_{k+1}} u_N^{(k+1)}(\tau_{k+1})\right\|_{L^2} d\tau_{k+1}.$$

Without loss of generality, set $\mu_m(k+1)=1$ and look at $\tilde{B}^1_{N,1,k+1,\tau_{k+1}}$, we have

$$\begin{aligned}
& \int \left| R^{(k)}\tilde{B}^1_{N,1,k+1,\tau_{k+1}} u_N^{(k+1)}(\tau_{k+1})\right|^2 d\mathbf{y}_k d\mathbf{y}'_k \\
=\ & \int \left| R^{(k)} \int \tilde{V}_{N,\tau_{k+1}}(y_1-y_{k+1}) u_N^{(k+1)}(\tau_{k+1},\mathbf{y}_k,y_{k+1};\mathbf{y}'_k,y_{k+1})dy_{k+1}\right|^2 d\mathbf{y}_k d\mathbf{y}'_k \\
\leqslant\ & C\int \left| \int \left(\tilde{V}_{N,\tau_{k+1}}\right)'(y_1-y_{k+1}) \left(\prod_{j=2}^k |\nabla_{y_j}|\right)\left(\prod_{j=1}^k \left|\nabla_{y'_j}\right|\right) u_N^{(k+1)}(\tau_{k+1},\mathbf{y}_k,y_{k+1};\mathbf{y}'_k,y_{k+1})dy_{k+1}\right|^2 d\mathbf{y}_k d\mathbf{y}'_k \\
& +C\int \left| \int \tilde{V}_{N,\tau_{k+1}}(y_1-y_{k+1}) R^{(k)} u_N^{(k+1)}(\tau_{k+1},\mathbf{y}_k,y_{k+1};\mathbf{y}'_k,y_{k+1})dy_{k+1}\right|^2 d\mathbf{y}_k d\mathbf{y}'_k \\
=\ & C(I+II).
\end{aligned}$$

Noticing that $\left\|\tilde{V}_\tau\right\|_{H^2} \leqslant C\|V\|_{H^2}$ uniformly for $\tau\in[0,\frac{\tan\omega T_0}{\omega}]$, we can then estimate

$$\begin{aligned}
& I \\
=\ & \int \left| \int \left(\tilde{V}_{N,\tau_{k+1}}\right)'(y_1-y_{k+1}) \left(\prod_{j=2}^k |\nabla_{y_j}|\right)\left(\prod_{j=1}^k \left|\nabla_{y'_j}\right|\right) u_N^{(k+1)}(\tau_{k+1},\mathbf{y}_k,y_{k+1};\mathbf{y}'_k,y_{k+1})dy_{k+1}\right|^2 d\mathbf{y}_k d\mathbf{y}'_k \\
\leqslant\ & \int d\mathbf{y}_k d\mathbf{y}'_k \left(\int \left|\left(\tilde{V}_{N,\tau_{k+1}}\right)'(y_1-y_{k+1})\right|^2 dy_{k+1}\right) \\
& \times \left(\int \left| \left(\prod_{j=2}^k |\nabla_{y_j}|\right)\left(\prod_{j=1}^k \left|\nabla_{y'_j}\right|\right) u_N^{(k+1)}(\tau_{k+1},\mathbf{y}_k,y_{k+1};\mathbf{y}'_k,y_{k+1})\right|^2 dy_{k+1}\right) \quad \text{(Cauchy-Schwarz)} \\
\leqslant\ & CN^{5\beta}\left\|V'\right\|^2_{L^2} \int \left(\int \left| \left(\prod_{j=2}^k |\nabla_{y_j}|\right)\left(\prod_{j=1}^k \left|\nabla_{y'_j}\right|\right) u_N^{(k+1)}(\tau_{k+1},\mathbf{y}_k,y_{k+1};\mathbf{y}'_k,y_{k+1})\right|^2 dy_{k+1}\right) d\mathbf{y}_k d\mathbf{y}'_k \\
\leqslant\ & CN^{5\beta}\left\|V'\right\|^2_{L^2} \int d\mathbf{y}_k d\mathbf{y}'_k \\
& \times \left(\int \left| (1-\triangle_{y_{k+1}})^{\frac{1}{2}}\left(1-\triangle_{y'_{k+1}}\right)^{\frac{1}{2}} \left(\prod_{j=2}^k |\nabla_{y_j}|\right)\left(\prod_{j=1}^k \left|\nabla_{y'_j}\right|\right) u_N^{(k+1)}(\tau_{k+1},\mathbf{y}_k,y_{k+1};\mathbf{y}'_k,y'_{k+1})\right|^2 dy_{k+1}dy'_{k+1}\right) \\
& \text{(Trace Theorem)} \\
\leqslant\ & CN^{5\beta}\left\|V'\right\|^2_{L^2} C^{k+1} \quad \text{(Condition 4.4)}
\end{aligned}$$

and

$$\begin{aligned}
II\ =\ & \int \left| \int \tilde{V}_{N,\tau_{k+1}}(y_1-y_{k+1}) R^{(k)} u_N^{(k+1)}(\tau_{k+1},\mathbf{y}_k,y_{k+1};\mathbf{y}'_k,y_{k+1})dy_{k+1}\right|^2 d\mathbf{y}_k d\mathbf{y}'_k \\
\leqslant\ & CN^{3\beta}\|V\|^2_{L^2} C^{k+1} \quad \text{(Same method as } I\text{)}.
\end{aligned}$$

Accordingly,

$$\int \left| R^{(k)} \tilde{B}^{1}_{N,1,k+1,\tau_{k+1}} u_N^{(k+1)}(\tau_{k+1}) \right|^2 d\mathbf{y}_k d\mathbf{y}'_k \leqslant CN^{5\beta} \|V\|_{H^2}^2 C^{k+1}.$$

Remark 4. *The estimates of I and II may not be optimal. But they are good enough for proving estimate 6.5 for arbitrary $\beta > 0$.*

Thence

$$\begin{aligned} & \int_0^T \left\| R^{(1)} \tilde{B}_{N,1,2.\tau_2} InteractionPart^{(k)}(\tau_2) \right\|_{L^2} d\tau_2 \\ \leqslant & \sum_m C(CT^{\frac{1}{2}})^{k-1} \int_0^T \left\| R^{(k)} \tilde{B}_{N,\mu_m(k+1),k+1,\tau_{k+1}} u_N^{(k+1)}(\tau_{k+1}) \right\|_{L^2} d\tau_{k+1}. \\ \leqslant & 4^{k-1} C(CT^{\frac{1}{2}})^{k-1} T^{\frac{1}{2}} \left(CN^{\frac{5\beta}{2}} \|V\|_{H^2} C^{k+1} \right) \\ \leqslant & C \|V\|_{H^2} (T^{\frac{1}{2}})^k N^{\frac{5\beta}{2}} C^k. \end{aligned}$$

Take the coupling level $k = \ln N$, we have

$$\int_0^T \left\| R^{(1)} \tilde{B}_{N,1,2.\tau_2} InteractionPart^{(k)}(\tau_2) \right\|_{L^2} d\tau_2 \leqslant C \|V\|_{H^2}^2 (T^{\frac{1}{2}})^{\ln N} N^{\frac{5\beta}{2}} N^c.$$

Selecting T such that

$$T \leqslant e^{-(5\beta+2C)}$$

ensures that

$$(T^{\frac{1}{2}})^{\ln N} N^{\frac{5\beta}{2}} N^c \leqslant 1$$

and thence

$$\int_0^T \left\| R^{(1)} \tilde{B}_{N,1,2.\tau_2} InteractionPart^{(k)}(\tau_2) \right\|_{L^2} d\tau_2 \leqslant C$$

where C is independent of N.

We remind the reader that, at this point, we have obtained estimates 6.3, 6.4, and 6.5 for a sufficiently small T determined by the controlling constant in condition 4.4 and independent of N. Thus one can repeat the argument to acquire the estimates for any finite time $T \in \left[0, \frac{\tan \omega T_0}{\omega}\right]$ through bootstrapping and condition 4.4. Whence, we have earned estimate 6.1 and established Theorem 5. The rest of this section is the proof of Proposition 3.

6.0.4. *Proof of Proposition 3 .* We will utilize the lemma.

Lemma 10. [18]

$$\int V(x_1 - x_2) |f(x_1, x_2)|^2 dx_1 dx_2 \leqslant C \|V\|_{L^1} \int \left| (1 - \triangle_{x_1})^{\frac{1}{2}} (1 - \triangle_{x_2})^{\frac{1}{2}} f(x_1, x_2) \right|^2 dx_1 dx_2.$$

In particular,

$$\int |V(x_1 - x_2)|^2 |f(x_1, x_2)|^2 dx_1 dx_2 \leqslant C \|V\|_{L^2}^2 \int \left| (1 - \triangle_{x_1})^{\frac{1}{2}} (1 - \triangle_{x_2})^{\frac{1}{2}} f(x_1, x_2) \right|^2 dx_1 dx_2.$$

Proof. This is Lemma A.3 in [18]. □

Without loss of generality, we show Proposition 3 for $k = 2$ which corresponds to

$$\int_0^T \left\| \left(|\nabla_{y_1}| |\nabla_{y_2}| |\nabla_{y'_1}| |\nabla_{y'_2}| \right) \left(\frac{\tilde{V}_{N,\tau}(y_1 - y_2)}{N} u_N^{(2)}(\tau, \mathbf{y}_2; \mathbf{y}'_2) \right) \right\|_{L^2} d\tau \leqslant C_0 C^2 T \tag{6.7}$$

and

$$\int_0^T \left\| \left(|\nabla_{y_1}| |\nabla_{y_2}| |\nabla_{y'_1}| |\nabla_{y'_2}| \right) \left(\frac{\tilde{V}_{N,\tau}(y'_1 - y'_2)}{N} u_N^{(2)}(\tau, \mathbf{y}_2; \mathbf{y}'_2) \right) \right\|_{L^2} d\tau \leqslant C_0 C^2 T.$$

By similarity we only prove estimate 6.7. For a general k, there are $2k^2$ terms in $\tilde{V}_{N,\tau}^{(k)}$, whence $R_N^{(k)} \tilde{V}_{N,\tau}^{(k)} \gamma_N^{(k)}$ has about $8k^2$ terms by Leibniz's rule. Since $8k^2$ can be absorbed into C^k, the method here applies.

First of all, $\beta \in \left(0, \frac{2}{7}\right]$ implies the following properties of $\tilde{V}_{N,\tau}/N$:

$$\begin{aligned}
\left\|\tilde{V}_{N,\tau}/N\right\|_{L^\infty} &= N^{3\beta-1}\left\|\tilde{V}_\tau\right\|_{L^\infty} < N^{-\frac{1}{7}}\left\|\tilde{V}_\tau\right\|_{L^\infty},\\
\left\|\left(\tilde{V}_{N,\tau}\right)'/N\right\|_{L^p} &= N^{4\beta-1-\frac{3\beta}{p}}\left\|\left(\tilde{V}_\tau\right)'\right\|_{L^p} = N^{-\left(\frac{6}{7p}-\frac{1}{7}+\left(4-\frac{3}{p}\right)\varepsilon_0\right)}\left\|\left(\tilde{V}_\tau\right)'\right\|_{L^p}, \text{ decays up to } p=6,\\
\left\|\left(\tilde{V}_{N,\tau}\right)''/N\right\|_{L^2} &\leqslant N^{-\frac{7}{2}\varepsilon_0}\left\|\left(\tilde{V}_\tau\right)''\right\|_{L^2}.
\end{aligned}$$

where ε_0 is $\left(\frac{2}{7}-\beta\right) \geqslant 0$. On the one hand, $\left\|\tilde{V}_\tau\right\|_{H^2} \leqslant C\|V\|_{H^2}$ uniformly for $\tau \in \left[0, \frac{\tan\omega T_0}{\omega}\right]$. On the other hand we assume V is a nonnegative $L^1(\mathbb{R}^3)\cap H^2(\mathbb{R}^3)$ function. Thus we know $\tilde{V}_\tau \in L^\infty$ and $\left(\tilde{V}_\tau\right)' \in L^6$.

We compute

$$\begin{aligned}
&\left(|\nabla_{y_1}||\nabla_{y_2}||\nabla_{y_1'}|\left|\nabla_{y_2'}\right|\right)\left(N^{-1}\tilde{V}_{N,\tau}(y_1-y_2)u_N^{(2)}(\tau,\mathbf{y}_2;\mathbf{y}_2')\right)\\
=\;&\left(|\nabla_{y_1}||\nabla_{y_2}|N^{-1}\tilde{V}_{N,\tau}(y_1-y_2)\right)\left(|\nabla_{y_1'}|\left|\nabla_{y_2'}\right|u_N^{(2)}(\tau,\mathbf{y}_2;\mathbf{y}_2')\right)\\
&+\left(|\nabla_{y_1}|N^{-1}\tilde{V}_{N,\tau}(y_1-y_2)\right)\left(|\nabla_{y_2}||\nabla_{y_1'}|\left|\nabla_{y_2'}\right|u_N^{(2)}(\tau,\mathbf{y}_2;\mathbf{y}_2')\right)\\
&+\left(|\nabla_{y_2}|N^{-1}\tilde{V}_{N,\tau}(y_1-y_2)\right)\left(|\nabla_{y_1}||\nabla_{y_1'}|\left|\nabla_{y_2'}\right|u_N^{(2)}(\tau,\mathbf{y}_2;\mathbf{y}_2')\right)\\
&+N^{-1}\tilde{V}_{N,\tau}(y_1-y_2)\left(|\nabla_{y_1}||\nabla_{y_2}||\nabla_{y_1'}|\left|\nabla_{y_2'}\right|\right)u_N^{(2)}(\tau,\mathbf{y}_2;\mathbf{y}_2').
\end{aligned}$$

But

$$\begin{aligned}
&\sup_\tau\left\|\left(|\nabla_{y_1}||\nabla_{y_2}|N^{-1}\tilde{V}_{N,\tau}(y_1-y_2)\right)\left(|\nabla_{y_1'}|\left|\nabla_{y_2'}\right|u_N^{(2)}(\tau,\mathbf{y}_2;\mathbf{y}_2')\right)\right\|_{L^2}^2\\
=\;&\sup_\tau\int\left|N^{-1}\left(\tilde{V}_{N,\tau}\right)''(y_1-y_2)\right|^2\left||\nabla_{y_1'}|\left|\nabla_{y_2'}\right|u_N^{(2)}(\tau,\mathbf{y}_2;\mathbf{y}_2')\right|^2 d\mathbf{y}_2 d\mathbf{y}_2'\\
\leqslant\;&\sup_\tau\int\left(\int\left|N^{-1}\left(\tilde{V}_{N,\tau}\right)''(y)\right|^2 dy\right)\left(\int\left|(1-\triangle_{y_1})^{\frac{1}{2}}(1-\triangle_{y_2})^{\frac{1}{2}}|\nabla_{y_1'}|\left|\nabla_{y_2'}\right|u_N^{(2)}(\tau,\mathbf{y}_2;\mathbf{y}_2')\right|^2 d\mathbf{y}_2\right)d\mathbf{y}_2'\\
&\text{(Lemma 10)}\\
\leqslant\;&CN^{-7\varepsilon_0}\|V''\|_{L^2}^2\sup_\tau\int\left|(1-\triangle_{y_1})^{\frac{1}{2}}(1-\triangle_{y_2})^{\frac{1}{2}}\left(1-\triangle_{y_1'}\right)^{\frac{1}{2}}\left(1-\triangle_{y_2'}\right)^{\frac{1}{2}}u_N^{(2)}(\tau,\mathbf{y}_2;\mathbf{y}_2')\right|^2 d\mathbf{y}_2 d\mathbf{y}_2'\\
\leqslant\;&N^{-7\varepsilon_0}\|V''\|_{L^2}^2 C^2 \text{ (Condition 4.4)}
\end{aligned}$$

and

$$\begin{aligned}
&\sup_\tau\left\|N^{-1}\tilde{V}_{N,\tau}(y_1-y_2)\left(|\nabla_{y_1}||\nabla_{y_2}||\nabla_{y_1'}|\left|\nabla_{y_2'}\right|\right)u_N^{(2)}(\tau,\mathbf{y}_2;\mathbf{y}_2')\right\|_{L^2}^2\\
=\;&\sup_\tau\int\left|N^{-1}\tilde{V}_{N,\tau}(y_1-y_2)\right|^2\left|\left(|\nabla_{y_1}||\nabla_{y_2}||\nabla_{y_1'}|\left|\nabla_{y_2'}\right|\right)u_N^{(2)}(\tau,\mathbf{y}_2;\mathbf{y}_2')\right|^2 d\mathbf{y}_2 d\mathbf{y}_2'\\
\leqslant\;&\sup_\tau\int\left\|N^{-1}\tilde{V}_{N,\tau}\right\|_{L^\infty}^2\left|(1-\triangle_{y_1})^{\frac{1}{2}}(1-\triangle_{y_2})^{\frac{1}{2}}\left(1-\triangle_{y_1'}\right)^{\frac{1}{2}}\left(1-\triangle_{y_2'}\right)^{\frac{1}{2}}u_N^{(2)}(\tau,\mathbf{y}_2;\mathbf{y}_2')\right|^2 d\mathbf{y}_2 d\mathbf{y}_2'\\
\leqslant\;&N^{-\frac{2}{7}}\|V\|_{L^\infty}^2\sup_\tau\int\left|(1-\triangle_{y_1})^{\frac{1}{2}}(1-\triangle_{y_2})^{\frac{1}{2}}\left(1-\triangle_{y_1'}\right)^{\frac{1}{2}}\left(1-\triangle_{y_2'}\right)^{\frac{1}{2}}u_N^{(2)}(\tau,\mathbf{y}_2;\mathbf{y}_2')\right|^2 d\mathbf{y}_2 d\mathbf{y}_2'\\
\leqslant\;&CN^{-\frac{2}{7}}\|V\|_{H^2}^2 C^2 \text{ (Sobolev and Condition 4.4).}
\end{aligned}$$

Since the same method applies to the middle terms, we have obtained estimate 6.7 and hence Proposition 3.

7. Conclusion

In this paper, we have rigorously derived the 3D cubic nonlinear Schrödinger equation with a quadratic trap from the N-body linear Schrödinger equation. The main novelty is that we allow a quadratic trap in our analysis and the main technical improvements are the simplified proof of the Klainerman-Machedon space-time bound as compared to non-trap case in Chen and Pavlović [7], and the extension of the range of β from $(0, 1/4)$ in Chen and Pavlović [7] to $(0, 2/7]$. Compared to the 2D work [12] which is also by the author, the 3D problem in this paper is of critical regularity. To explain what we mean by critical: in 2D one easily obtains the $|\nabla|^{\frac{1}{2}}$-space-time bound needed for the uniqueness theorem by a trace theorem; in 3D the only way to obtain the space-time bound 4.5 is through smoothing estimates since one does not have

enough regularity to apply a trace theorem. Thence the key arguments in 3D are more involved and totally different from the 2D case which is a subcritical problem. Moreover, we have established the trace norm convergence in the main theorem which is a stronger result than the Hilbert-Schmidt norm convergence.

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9. Appendix I: Proof of Corollary 1

For the purpose of this Appdendix I, we may assume $\omega = 0$. Or in other words, we skip Steps I and IV of the proof of Theorem 4 here. When the desired limit is an orthogonal projection, one does not need this proof. This is a functional analysis argument and all operators mentioned in this Appendix I acts on $L^2\left(\mathbb{R}^{3k}\right)$. We prove Corollary 1 by verifying the hypothesis of the following lemma.

Lemma 11. [35] *Assume the operator sequence $\{A_n\}$ satisfies that, as bounded operators, $A_n \rightharpoonup A$, $A_n^* \rightharpoonup A^*$ and $|A_n| \rightharpoonup |A|$ in the weak sense. If*

$$\lim_{N\to\infty} \operatorname{Tr} |A_n| = \operatorname{Tr} |A|,$$

then

$$\lim_{N\to\infty} \operatorname{Tr} |A_n - A| = 0.$$

Proof. This is Theorem 2.20 in [35]. It implies the Grümm's convergence theorem (Theorem 2.19 of [35]) used in [21]. ☐

We first observe that condition 1.6 implies the a-priori estimate

$$\sup_{t\in[0,T]} \operatorname{Tr} \left(\prod_{j=1}^{k} \left(1 - \triangle_{x_j}\right) \right) \gamma_N^{(k)} \leqslant C^k.$$

Thus we have the compactness argument and the uniqueness argument to conclude that, as trace class operator kernels,

$$\gamma_N^{(k)}(t, \mathbf{x}_k; \mathbf{x}_k') \rightharpoonup \gamma^{(k)}(t, \mathbf{x}_k; \mathbf{x}_k') \text{ (weak*)}.$$

Remark 5. *Because we assume $\omega = 0$, both of the Erdös-Schlein-Yau uniqueness theorem* [18] *and the Klainerman-Machedon uniqueness theorem* [30] *apply here. For the general case, one has to use the main argument in this paper.*

The above weak* convergence as trace class operator kernels infers that as Hilbert-Schmidt kernels,

$$\gamma_N^{(k)}(t, \mathbf{x}_k; \mathbf{x}_k') \rightharpoonup \gamma^{(k)}(t, \mathbf{x}_k; \mathbf{x}_k') \text{ (weak*)},$$

because there are less test functions. Since $L^2(d\mathbf{x}_k d\mathbf{x}_k')$ is reflexive, the weak* convergence is no different from the weak convergence. Thus we know that as Hilbert-Schmidt kernels and hence as bounded operator kernels,

$$\gamma_N^{(k)}(t, \mathbf{x}_k; \mathbf{x}_k') \rightharpoonup \gamma^{(k)}(t, \mathbf{x}_k; \mathbf{x}_k') \text{ (weak)}.$$

At this point, we have verified that, as bounded operators, $A_n \rightharpoonup A$ and $A_n^* \rightharpoonup A^*$ in the weak sense since $\gamma_N^{(k)}$ and $\gamma^{(k)}$ are self adjoint. We now check $|A_n| \rightharpoonup |A|$.

To check $|A_n| \rightharpoonup |A|$, one first notice that $\gamma_N^{(k)}$ and $\gamma^{(k)}$ has only real eigenvalues since they are self adjoint. Moreover, $\gamma_N^{(k)}(t, \mathbf{x}_k; \mathbf{x}_k')$ has no negative eigenvalues by definition 1.3, in fact,

$$\int \left(\int \left(\int \phi(x,z)\bar{\phi}(y,z)dz \right) f(y)dy \right) \bar{f}(x)dx = \int dz \left| \int \phi(x,z)\bar{f}(x)dx \right|^2 \geqslant 0.$$

Notice that $f(\mathbf{x}_k)\bar{f}(\mathbf{x}_k')$ in the estimate above is also a Hilbert-Schmidt kernel, the definition of weak convergence in $L^2(d\mathbf{x}_k d\mathbf{x}_k')$ then gives

$$\int \gamma^{(k)}(t,\mathbf{x}_k;\mathbf{x}_k')\overline{f(\mathbf{x}_k)\bar{f}(\mathbf{x}_k')}d\mathbf{x}_k d\mathbf{x}_k' = \lim_{N\to\infty}\int \gamma_N^{(k)}(t,\mathbf{x}_k;\mathbf{x}_k')\overline{f(\mathbf{x}_k)\bar{f}(\mathbf{x}_k')}d\mathbf{x}_k d\mathbf{x}_k' \geqslant 0,$$

as $\gamma_N^{(k)}(t,\mathbf{x}_k;\mathbf{x}_k') \rightharpoonup \gamma^{(k)}(t,\mathbf{x}_k;\mathbf{x}_k')$ weakly in $L^2(d\mathbf{x}_k d\mathbf{x}_k')$. So we have checked $|A_n| \rightharpoonup |A|$ because $|A_n| = A_n$ and $|A| = A$.

To prove Corollary 1, by Lemma 11, it remains to argue

$$\lim_{N\to\infty}\operatorname{Tr}\left|\gamma_N^{(k)}(t,\mathbf{x}_k;\mathbf{x}_k')\right| = \operatorname{Tr}\left|\gamma^{(k)}(t,\mathbf{x}_k;\mathbf{x}_k')\right|.$$

Notice that we have the conservation of trace

$$\begin{aligned}\int \gamma_N^{(k)}(t,\mathbf{x}_k;\mathbf{x}_k)d\mathbf{x}_k &= \int \gamma_N^{(k)}(0,\mathbf{x}_k;\mathbf{x}_k)d\mathbf{x}_k \text{ (By definition 1.3)}\\ \int \gamma^{(k)}(t,\mathbf{x}_k;\mathbf{x}_k)d\mathbf{x}_k &= \int \gamma^{(k)}(0,\mathbf{x}_k;\mathbf{x}_k)d\mathbf{x}_k \text{ [8].}\end{aligned}$$

and we have shown that $\gamma_N^{(k)}(t,\mathbf{x}_k;\mathbf{x}_k')$ and $\gamma^{(k)}(t,\mathbf{x}_k;\mathbf{x}_k')$ have no negative eigenvalues, hence

$$\begin{aligned}\operatorname{Tr}\left|\gamma_N^{(k)}(t,\mathbf{x}_k;\mathbf{x}_k')\right| &= \int \gamma_N^{(k)}(t,\mathbf{x}_k;\mathbf{x}_k)d\mathbf{x}_k = \int \gamma_N^{(k)}(0,\mathbf{x}_k;\mathbf{x}_k)d\mathbf{x}_k,\\ \operatorname{Tr}\left|\gamma^{(k)}(t,\mathbf{x}_k;\mathbf{x}_k')\right| &= \int \gamma^{(k)}(t,\mathbf{x}_k;\mathbf{x}_k)d\mathbf{x}_k = \int \gamma^{(k)}(0,\mathbf{x}_k;\mathbf{x}_k)d\mathbf{x}_k.\end{aligned}$$

On the one hand,

$$\int \gamma^{(k)}(0,\mathbf{x}_k;\mathbf{x}_k)d\mathbf{x}_k = \operatorname{Tr}\left|\gamma^{(k)}(0,\mathbf{x}_k;\mathbf{x}_k')\right| = \lim_{N\to\infty}\operatorname{Tr}\left|\gamma_N^{(k)}(0,\mathbf{x}_k;\mathbf{x}_k')\right| = \lim_{N\to\infty}\operatorname{Tr}\left|\gamma_N^{(k)}(t,\mathbf{x}_k;\mathbf{x}_k')\right|,$$

on the other hand,

$$\operatorname{Tr}\left|\gamma^{(k)}(t,\mathbf{x}_k;\mathbf{x}_k')\right| = \int \gamma^{(k)}(t,\mathbf{x}_k;\mathbf{x}_k)d\mathbf{x}_k = \int \gamma^{(k)}(0,\mathbf{x}_k;\mathbf{x}_k)d\mathbf{x}_k.$$

That is

$$\operatorname{Tr}\left|\gamma^{(k)}(t,\mathbf{x}_k;\mathbf{x}_k')\right| = \lim_{N\to\infty}\operatorname{Tr}\left|\gamma_N^{(k)}(t,\mathbf{x}_k;\mathbf{x}_k')\right|.$$

Whence we conclude the proof of Corollary 1 by Lemma 11.

10. Appendix II: Proof of Theorem 7

In this Appendix II, instead of proving Theorem 7, we establish the following general estimate. On the one hand, the special case (Theorem 7) does not yield a shorter proof. Unlike estimate A.8 in [7], Theorem 7 does not follow directly from [29, 30]. The time dependence of the potential makes the proof closer to the Beals-Bezard simplification [2] of the Klainerman-Machedon null form paper [29]. On the other hand, the anisotropic version of the main theorem (Theorem 3) requires the following theorem.

Theorem 8. *Define the diagonal matrices*

$$L_t = \begin{pmatrix} a_1(t) & 0 & 0\\ 0 & a_2(t) & 0\\ 0 & 0 & a_3(t)\end{pmatrix} \text{ and } B_t = \begin{pmatrix} \beta_1(t) & 0 & 0\\ 0 & \beta_2(t) & 0\\ 0 & 0 & \beta_3(t)\end{pmatrix},$$

where $a_l \in L^1_{loc}(\mathbb{R})$ functions and

$$a_l, \beta_l \geqslant c_0 > 0 \text{ a.e.}$$

Suppose $u(t,x_1,x_2,x_2')$ solves the Schrödinger equation

$$\begin{aligned} iu_t + \operatorname{div}_{x_1}(L_t\nabla_{x_1})u + \operatorname{div}_{x_2}(L_t\nabla_{x_2})u - \operatorname{div}_{x_2'}\left(L_t\nabla_{x_2'}\right)u &= 0 \text{ in } \mathbb{R}^{9+1} \\ u(0,x_1,x_2,x_2') &= f(x_1,x_2,x_2'),\end{aligned} \tag{10.1}$$

then there is a C independent of N and u such that

$$\begin{aligned}&\int_{\mathbb{R}^{3+1}}\left||\nabla_x|\left(\int V_N^t(x-x_2)\delta(x_2-x_2')u(t,x,x_2;x_2')dx_2dx_2'\right)\right|^2 dxdt\\ \leqslant\ & Cb_0^2\left\||\nabla_{x_1}||\nabla_{x_2}|\left|\nabla_{x_2'}\right|f\right\|_2^2,\end{aligned}$$

where

$$V_N^t(x) = \det(B_t)\,V_N(B_tx),$$

which has the property that

$$\sup_{t,\xi,N}\left|\widehat{V_N^t}(\xi)\right| \leqslant \int\left|V_N^t(x)\right|dx = b_0.$$

Through out this paper, we have used x to denote a 3D variable. However, the anisotropic feature of Theorem 8 requires us to use the individual components of a 3D vector in the proof. So we use bold symbols to represent a 3D vector in this Appendix II. e.g. $\mathbf{x} = (x_1, x_2, x_3)\in\mathbb{R}^3$.

We will make use of the lemma.

Lemma 12. [30] *Let* $\boldsymbol{\xi} \in \mathbb{R}^3$ *and* P *be a 2d plane or sphere in* $\mathbb{R}^3$ *with the usual induced surface measure* dS*.*

(1) Suppose $0 < a,\ b < 2,\ a + b > 2$*, then*

$$\int_P \frac{dS(\boldsymbol{\eta})}{|\boldsymbol{\xi}-\boldsymbol{\eta}|^a\,|\boldsymbol{\eta}|^b} \leqslant \frac{C}{|\boldsymbol{\xi}|^{a+b-2}}.$$

(2) Suppose $\varepsilon = \frac{1}{10}$*, then*

$$\int_P \frac{dS(\boldsymbol{\eta})}{\left|\frac{\boldsymbol{\xi}}{2}-\boldsymbol{\eta}\right|\,|\boldsymbol{\xi}-\boldsymbol{\eta}|^{2-\varepsilon}\,|\boldsymbol{\eta}|^{2-\varepsilon}} \leqslant \frac{C}{|\boldsymbol{\xi}|^{3-2\varepsilon}}.$$

Both constants in the above estimates are independent of P*.*

Proof. See pages 174 - 175 of [30]. □

By duality, to gain Theorem 8, it suffices to prove that

$$\begin{aligned}&\left|\int_{\mathbb{R}^{3+1}} h(t,\mathbf{x})\,|\nabla_{\mathbf{x}}|\left(\int V_N^t(\mathbf{x}-\mathbf{x}_2)\delta(\mathbf{x}_2{-}\mathbf{x}_2')u(t,\mathbf{x},\mathbf{x}_2;\mathbf{x}_2')d\mathbf{x}_2 d\mathbf{x}_2'\right)d\mathbf{x}dt\right|\\ \leqslant\ & Cb_0\left\|h\right\|_2\left\|\nabla_{\mathbf{x}_1}\nabla_{\mathbf{x}_2}\nabla_{\mathbf{x}_2'}f\right\|_2.\end{aligned}$$

For this purpose, let

$$A_t = \begin{pmatrix}\int_0^t a_1(s)ds & 0 & 0\\ 0 & \int_0^t a_2(s)ds & 0\\ 0 & 0 & \int_0^t a_3(s)ds\end{pmatrix}.$$

Consequently, the solution of equation 10.1 is recast to

$$u(t,\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_2') = \int e^{i(\boldsymbol{\xi}_1^T A_t\boldsymbol{\xi}_1+\boldsymbol{\xi}_2^T A_t\boldsymbol{\xi}_2\pm(\boldsymbol{\xi}_2')^T A_t\boldsymbol{\xi}_2')}e^{i\mathbf{x}_1\boldsymbol{\xi}_1}e^{i\mathbf{x}_2\boldsymbol{\xi}_2}e^{i\mathbf{x}_2'\boldsymbol{\xi}_2'}\hat{f}(\boldsymbol{\xi}_1,\boldsymbol{\xi}_2,\boldsymbol{\xi}_2')d\boldsymbol{\xi}_1 d\boldsymbol{\xi}_2 d\boldsymbol{\xi}_2',$$

which implies

$$\begin{aligned}&\int V_N^t(\mathbf{x}-\mathbf{x}_2)\delta(\mathbf{x}_2{-}\mathbf{x}_2')u(t,\mathbf{x},\mathbf{x}_2;\mathbf{x}_2')d\mathbf{x}_2 d\mathbf{x}_2'\\ =\ &\int \widehat{V_N^t}(\boldsymbol{\xi}_2)e^{i(\boldsymbol{\xi}_1^T A_t\boldsymbol{\xi}_1+\boldsymbol{\xi}_2^T A_t\boldsymbol{\xi}_2\pm(\boldsymbol{\xi}_2')^T A_t\boldsymbol{\xi}_2')}e^{i\mathbf{x}(\boldsymbol{\xi}_1+\boldsymbol{\xi}_2+\boldsymbol{\xi}_2')}\hat{f}(\boldsymbol{\xi}_1,\boldsymbol{\xi}_2,\boldsymbol{\xi}_2')d\boldsymbol{\xi}_1 d\boldsymbol{\xi}_2 d\boldsymbol{\xi}_2'.\end{aligned}$$

Accordingly, the spatial Fourier transform of

$$|\nabla_{\mathbf{x}}|\left(\int V_N^t(\mathbf{x}-\mathbf{x}_2)\delta(\mathbf{x}_2{-}\mathbf{x}_2')u(t,\mathbf{x},\mathbf{x}_2;\mathbf{x}_2')d\mathbf{x}_2 d\mathbf{x}_2'\right)$$

is

$$|\boldsymbol{\xi}_1|\int \widehat{V_N^t}(\boldsymbol{\xi}_2)e^{i((\boldsymbol{\xi}_1-\boldsymbol{\xi}_2-\boldsymbol{\xi}_2')^T A_t(\boldsymbol{\xi}_1-\boldsymbol{\xi}_2-\boldsymbol{\xi}_2')+\boldsymbol{\xi}_2^T A_t\boldsymbol{\xi}_2\pm(\boldsymbol{\xi}_2')^T A_t\boldsymbol{\xi}_2')}\hat{f}(\boldsymbol{\xi}_1-\boldsymbol{\xi}_2-\boldsymbol{\xi}_2',\boldsymbol{\xi}_2,\boldsymbol{\xi}_2')d\boldsymbol{\xi}_2 d\boldsymbol{\xi}_2',$$

which allows us to compute

$$\begin{aligned}
&\left|\int_{\mathbb{R}^{3+1}} h(t,\mathbf{x})\left|\nabla_{\mathbf{x}}\right|\left(\int V_N^t(\mathbf{x}-\mathbf{x}_2)\delta(\mathbf{x}_2-\mathbf{x}_2')u(t,\mathbf{x},\mathbf{x}_2;\mathbf{x}_2')d\mathbf{x}_2d\mathbf{x}_2'\right)d\mathbf{x}dt\right|^2\\
=&\left|\int |\boldsymbol{\xi}_1|\,\widehat{V_N^t}(\boldsymbol{\xi}_2)e^{i\left((\boldsymbol{\xi}_1-\boldsymbol{\xi}_2-\boldsymbol{\xi}_2')^TA_t(\boldsymbol{\xi}_1-\boldsymbol{\xi}_2-\boldsymbol{\xi}_2')+\boldsymbol{\xi}_2^TA_t\boldsymbol{\xi}_2\pm(\boldsymbol{\xi}_2')^TA_t\boldsymbol{\xi}_2'\right)}\right.\\
&\left.\hat f(\boldsymbol{\xi}_1-\boldsymbol{\xi}_2-\boldsymbol{\xi}_2',\boldsymbol{\xi}_2,\boldsymbol{\xi}_2')\hat h(t,\boldsymbol{\xi}_1)dtd\boldsymbol{\xi}_1d\boldsymbol{\xi}_2d\boldsymbol{\xi}_2'\right|^2 \text{ (spatial Fourier transform on } h)\\
=&\left|\int\left(\int \widehat{V_N^t}(\boldsymbol{\xi}_2)\,|\boldsymbol{\xi}_1|\,e^{i\left((\boldsymbol{\xi}_1-\boldsymbol{\xi}_2-\boldsymbol{\xi}_2')^TA_t(\boldsymbol{\xi}_1-\boldsymbol{\xi}_2-\boldsymbol{\xi}_2')+\boldsymbol{\xi}_2^TA_t\boldsymbol{\xi}_2\pm(\boldsymbol{\xi}_2')^TA_t\boldsymbol{\xi}_2'\right)}\hat h(t,\boldsymbol{\xi}_1)dt\right)\right.\\
&\left.\hat f(\boldsymbol{\xi}_1-\boldsymbol{\xi}_2-\boldsymbol{\xi}_2',\boldsymbol{\xi}_2,\boldsymbol{\xi}_2')d\boldsymbol{\xi}_1d\boldsymbol{\xi}_2d\boldsymbol{\xi}_2'\right|^2\\
\leqslant\; & I(h)\left\|\nabla_{\mathbf{x}_1}\nabla_{\mathbf{x}_2}\nabla_{\mathbf{x}_2'}f\right\|_{L^2}^2 \text{ (Cauchy-Schwarz)}
\end{aligned}$$

where

$$I(h)=\int\frac{|\boldsymbol{\xi}_1|^2\left|\int \widehat{V_N^t}(\boldsymbol{\xi}_2)e^{i\left((\boldsymbol{\xi}_1-\boldsymbol{\xi}_2-\boldsymbol{\xi}_2')^TA_t(\boldsymbol{\xi}_1-\boldsymbol{\xi}_2-\boldsymbol{\xi}_2')+\boldsymbol{\xi}_2^TA_t\boldsymbol{\xi}_2\pm(\boldsymbol{\xi}_2')^TA_t\boldsymbol{\xi}_2'\right)}\hat h(t,\boldsymbol{\xi}_1)dt\right|^2}{\left|\boldsymbol{\xi}_1-\boldsymbol{\xi}_2-\boldsymbol{\xi}_2'\right|^2|\boldsymbol{\xi}_2|^2\left|\boldsymbol{\xi}_2'\right|^2}d\boldsymbol{\xi}_1d\boldsymbol{\xi}_2d\boldsymbol{\xi}_2'.$$

So our purpose in the remainder of this Appendix II is to show that

$$I(h)\leqslant Cb_0^2\|h\|_{L^2}^2\,.$$

Noticing that, away from the factor $\widehat{V_N^t}(\boldsymbol{\xi}_2)$, the integral $I(h)$ is symmetric in $\left|\boldsymbol{\xi}_1-\boldsymbol{\xi}_2-\boldsymbol{\xi}_2'\right|$ and $|\boldsymbol{\xi}_2|$, it suffices that we deal with the region: $\left|\boldsymbol{\xi}_1-\boldsymbol{\xi}_2-\boldsymbol{\xi}_2'\right|>|\boldsymbol{\xi}_2|$ only since our proof treats $\widehat{V_N^t}(\boldsymbol{\xi}_2)$ as a harmless factor. We separate this region into two parts, Cases I and II.

Away from the region $\left|\boldsymbol{\xi}_1-\boldsymbol{\xi}_2-\boldsymbol{\xi}_2'\right|>|\boldsymbol{\xi}_2|$, there are other restrictions on the integration regions in Cases I and II. We state the restrictions in the beginning of both Cases I and II. Due to the limited space near "$\int$", we omit the actual region. The alert reader should bear this mind.

10.0.5. *Case I: $I(h)$ restricted to the region $\left|\boldsymbol{\xi}_2'\right|<|\boldsymbol{\xi}_2|$ with integration order $d\boldsymbol{\xi}_2$ prior to $d\boldsymbol{\xi}_2'$.* Write the phase function of the dt integral inside $I(h)$ as

$$\begin{aligned}
&\left(\boldsymbol{\xi}_1-\boldsymbol{\xi}_2-\boldsymbol{\xi}_2'\right)^TA_t\left(\boldsymbol{\xi}_1-\boldsymbol{\xi}_2-\boldsymbol{\xi}_2'\right)+\boldsymbol{\xi}_2^TA_t\boldsymbol{\xi}_2\pm\left(\boldsymbol{\xi}_2'\right)^TA_t\boldsymbol{\xi}_2'\\
=&\frac{\left(\boldsymbol{\xi}_1-\boldsymbol{\xi}_2'\right)^TA_t\left(\boldsymbol{\xi}_1-\boldsymbol{\xi}_2'\right)}{2}+2\left(\boldsymbol{\xi}_2-\frac{\boldsymbol{\xi}_1-\boldsymbol{\xi}_2'}{2}\right)^TA_t\left(\boldsymbol{\xi}_2-\frac{\boldsymbol{\xi}_1-\boldsymbol{\xi}_2'}{2}\right)\pm\left(\boldsymbol{\xi}_2'\right)^TA_t\boldsymbol{\xi}_2'.
\end{aligned}$$

The change of variable

$$\boldsymbol{\xi}_{2,new}=\boldsymbol{\xi}_{2,old}-\frac{\boldsymbol{\xi}_1-\boldsymbol{\xi}_2'}{2}\tag{10.2}$$

leads to the expression

$$\begin{aligned}
I(h)=&\int\frac{|\boldsymbol{\xi}_1|^2\left|\int \widehat{V_N^t}(\boldsymbol{\xi}_2)e^{i\left(\frac{(\boldsymbol{\xi}_1-\boldsymbol{\xi}_2')^TA_t(\boldsymbol{\xi}_1-\boldsymbol{\xi}_2')}{2}+2\boldsymbol{\xi}_2^TA_t\boldsymbol{\xi}_2\pm(\boldsymbol{\xi}_2')^TA_t\boldsymbol{\xi}_2'\right)}\hat h(t,\boldsymbol{\xi}_1)dt\right|^2}{\left|\boldsymbol{\xi}_2-\frac{\boldsymbol{\xi}_1-\boldsymbol{\xi}_2'}{2}\right|^2\left|\boldsymbol{\xi}_2+\frac{\boldsymbol{\xi}_1-\boldsymbol{\xi}_2'}{2}\right|^2\left|\boldsymbol{\xi}_2'\right|^2}d\xi_1d\xi_2d\xi_2'\\
=&\int\frac{|\boldsymbol{\xi}_1|^2}{\left|\boldsymbol{\xi}_2-\frac{\boldsymbol{\xi}_1-\boldsymbol{\xi}_2'}{2}\right|^2\left|\boldsymbol{\xi}_2+\frac{\boldsymbol{\xi}_1-\boldsymbol{\xi}_2'}{2}\right|^2\left|\boldsymbol{\xi}_2'\right|^2}\widehat{V_N^t}(\boldsymbol{\xi}_2)e^{i\left(2\frac{(\boldsymbol{\xi}_1-\boldsymbol{\xi}_2')^TA_t(\boldsymbol{\xi}_1-\boldsymbol{\xi}_2')}{2}+2\boldsymbol{\xi}_2^TA_t\boldsymbol{\xi}_2\pm(\boldsymbol{\xi}_2')^TA_t\boldsymbol{\xi}_2'\right)}\\
&e^{-i\left(\frac{(\boldsymbol{\xi}_1-\boldsymbol{\xi}_2')^TA_{t'}(\boldsymbol{\xi}_1-\boldsymbol{\xi}_2')}{2}+2\boldsymbol{\xi}_2^TA_{t'}\boldsymbol{\xi}_2\pm(\boldsymbol{\xi}_2')^TA_{t'}\boldsymbol{\xi}_2'\right)}\hat h(t,\boldsymbol{\xi}_1)\overline{\widehat{V_N^{t'}}(\boldsymbol{\xi}_2)}\overline{\hat h(t',\boldsymbol{\xi}_1)}dtdt'd\boldsymbol{\xi}_1d\boldsymbol{\xi}_2d\boldsymbol{\xi}_2'\\
=&\int d\boldsymbol{\xi}_1\int J(\overline{\hat h})(t,\boldsymbol{\xi}_1)\hat h(t,\boldsymbol{\xi}_1)dt
\end{aligned}$$

where

$$\begin{aligned}
J(\overline{\hat h})(t,\boldsymbol{\xi}_1)=&\int\frac{|\boldsymbol{\xi}_1|^2e^{i2\boldsymbol{\xi}_2^TA_t\boldsymbol{\xi}_2}e^{-i2\boldsymbol{\xi}_2^TA_{t'}\boldsymbol{\xi}_2}\widehat{V_N^t}(\boldsymbol{\xi}_2)\overline{\widehat{V_N^{t'}}(\boldsymbol{\xi}_2)}}{\left|\boldsymbol{\xi}_2-\frac{\boldsymbol{\xi}_1-\boldsymbol{\xi}_2'}{2}\right|^2\left|\boldsymbol{\xi}_2+\frac{\boldsymbol{\xi}_1-\boldsymbol{\xi}_2'}{2}\right|^2\left|\boldsymbol{\xi}_2'\right|^2}\\
&e^{i\left(\frac{(\boldsymbol{\xi}_1-\boldsymbol{\xi}_2')^T(A_t-A_{t'})(\boldsymbol{\xi}_1-\boldsymbol{\xi}_2')}{2}\pm(\boldsymbol{\xi}_2')^T(A_t-A_{t'})\boldsymbol{\xi}_2'\right)}\overline{\hat h(t',\boldsymbol{\xi}_1)}dt'd\boldsymbol{\xi}_2d\boldsymbol{\xi}_2'.
\end{aligned}$$

Assume for the moment that

$$\int \left|J(\overline{\hat{h}})(t,\boldsymbol{\xi}_1)\right|^2 dt \leqslant Cb_0^2 \left\|\hat{h}(\cdot,\boldsymbol{\xi}_1)\right\|_{L_t^2}^2$$

with C independent of h or $\boldsymbol{\xi}_1$, then we deduce that

$$I(h) \leqslant Cb_0^2 \int d\boldsymbol{\xi}_1 \left\|\hat{h}(\cdot,\boldsymbol{\xi}_1)\right\|_{L_t^2}^2.$$

Hence we end Case I by this proposition.

Proposition 4.

$$\int |J(f)(t,\boldsymbol{\xi}_1)|^2 dt \leqslant Cb_0^2 \|f(\cdot,\boldsymbol{\xi}_1)\|_{L_t^2}^2$$

where C is independent of f or $\boldsymbol{\xi}_1$.

Remark 6. *To avoid confusing notation in the proof of the proposition, we use $f(t',\boldsymbol{\xi}_1)$ to replace $\overline{\hat{h}(t',\boldsymbol{\xi}_1)}$.*

Proof. Again, by duality, we just need to prove

$$\left|\int J(f)(t,\boldsymbol{\xi}_1)\overline{g(t)}dt\right| \leqslant C \|f(\cdot,\boldsymbol{\xi}_1)\|_{L_t^2} \|g\|_{L_t^2}.$$

For convenience, let

$$\phi(t,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2') = \frac{(\boldsymbol{\xi}_1-\boldsymbol{\xi}_2')^T A_t (\boldsymbol{\xi}_1-\boldsymbol{\xi}_2')}{2} \pm (\boldsymbol{\xi}_2')^T A_t \boldsymbol{\xi}_2'.$$

Then

$$\begin{aligned}
&\left|\int J(f)(t,\boldsymbol{\xi}_1)\overline{g(t)}dt\right| \\
=\ & \left|\int \frac{|\boldsymbol{\xi}_1|^2 e^{i2\boldsymbol{\xi}_2^T A_t \boldsymbol{\xi}_2} e^{-i2\boldsymbol{\xi}_2^T A_{t'} \boldsymbol{\xi}_2}}{\left|\boldsymbol{\xi}_2 - \frac{\boldsymbol{\xi}_1-\boldsymbol{\xi}_2'}{2}\right|^2 \left|\boldsymbol{\xi}_2 + \frac{\boldsymbol{\xi}_1-\boldsymbol{\xi}_2'}{2}\right|^2 |\boldsymbol{\xi}_2'|^2} \left(\overline{\widehat{V_N^{t'}}(\boldsymbol{\xi}_2)} e^{-i\phi(t',\boldsymbol{\xi}_1,\boldsymbol{\xi}_2')} f(t',\boldsymbol{\xi}_1)\right)\right. \\
& \left. \left(\widehat{V_N^t}(\boldsymbol{\xi}_2)\overline{e^{-i\phi(t,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2')}g(t)}\right) dt dt' d\boldsymbol{\xi}_2 d\boldsymbol{\xi}_2'\right| \\
=\ & \left|\int \frac{|\boldsymbol{\xi}_1|^2 d\boldsymbol{\xi}_2 d\boldsymbol{\xi}_2'}{\left|\boldsymbol{\xi}_2 - \frac{\boldsymbol{\xi}_1-\boldsymbol{\xi}_2'}{2}\right|^2 \left|\boldsymbol{\xi}_2 + \frac{\boldsymbol{\xi}_1-\boldsymbol{\xi}_2'}{2}\right|^2 |\boldsymbol{\xi}_2'|^2} \left(\int e^{2i\boldsymbol{\xi}_2^T A_t \boldsymbol{\xi}_2} \left(\widehat{V_N^t}(\boldsymbol{\xi}_2)\overline{e^{-i\phi(t,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2')}g(t)}\right) dt\right)\right. \\
& \left. \left(\int e^{-2i\boldsymbol{\xi}_2^T A_{t'} \boldsymbol{\xi}_2} \left(\overline{\widehat{V_N^{t'}}(\boldsymbol{\xi}_2)} e^{-i\phi(t',\boldsymbol{\xi}_1,\boldsymbol{\xi}_2')} f(t',\boldsymbol{\xi}_1)\right) dt'\right)\right| \\
\leqslant\ & \int \frac{|\boldsymbol{\xi}_1|^2 d\boldsymbol{\xi}_2'}{|\boldsymbol{\xi}_2'|^2} \int \frac{d\boldsymbol{\xi}_2}{\left|\boldsymbol{\xi}_2 - \frac{\boldsymbol{\xi}_1-\boldsymbol{\xi}_2'}{2}\right|^2 \left|\boldsymbol{\xi}_2 + \frac{\boldsymbol{\xi}_1-\boldsymbol{\xi}_2'}{2}\right|^2} \left|\int e^{2i\boldsymbol{\xi}_2^T A_t \boldsymbol{\xi}_2} \left(\widehat{V_N^t}(\boldsymbol{\xi}_2)\overline{e^{-i\phi(t,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2')}g(t)}\right) dt\right| \\
& \left|\int e^{-2i\boldsymbol{\xi}_2^T A_{t'} \boldsymbol{\xi}_2} \left(\overline{\widehat{V_N^{t'}}(\boldsymbol{\xi}_2)} e^{-i\phi(t',\boldsymbol{\xi}_1,\boldsymbol{\xi}_2')} f(t',\boldsymbol{\xi}_1)\right) dt'\right|
\end{aligned}$$

To deal with the dt and dt' integrals, for every fixed $\boldsymbol{\xi}_2$, let

$$u(t) = 2\frac{\boldsymbol{\xi}_2^T A_t \boldsymbol{\xi}_2}{|\boldsymbol{\xi}_2|^2}$$

then

$$\frac{du}{dt} = 2\frac{a_1(t)\xi_{2,1}^2 + a_2(t)\xi_{2,2}^2 + a_3(t)\xi_{2,3}^2}{|\boldsymbol{\xi}_2|^2} \geqslant 2c_0 > 0$$

which provides a well-defined inverse $t(u)$.

Consequently, the integral

$$\begin{aligned}
& \overline{\int e^{2i\boldsymbol{\xi}_2^T A_t \boldsymbol{\xi}_2} \left(\widehat{V_N^t}(\boldsymbol{\xi}_2)\overline{e^{-i\phi(t,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2')}g(t)}\right) dt} \\
=\ & \int e^{-iu|\boldsymbol{\xi}_2|^2} \left(\overline{\widehat{V_N^{t(u)}}(\boldsymbol{\xi}_2)} e^{-i\phi(t(u),\boldsymbol{\xi}_1,\boldsymbol{\xi}_2')} g(t(u)) \left|\frac{dt}{du}\right|\right) du
\end{aligned}$$

is indeed the Fourier transform of

$$G(u) = \overline{\widehat{V_N^{t(u)}}(\boldsymbol{\xi}_2)} e^{-i\phi(t(u),\boldsymbol{\xi}_1,\boldsymbol{\xi}_2')} g(t(u)) \left|\frac{dt}{du}\right|.$$

This is well-defined since

$$\begin{aligned}
\int_{\mathbb{R}} |G(u)|^2 \, du &= \int_{\mathbb{R}} \left| \widehat{V_N^{t(u)}}(\boldsymbol{\xi}_2) \right|^2 \left| \overline{e^{-i\phi(t(u),\boldsymbol{\xi}_1,\boldsymbol{\xi}_2)} g(t(u))} \left| \frac{dt}{du} \right| \right|^2 du \\
&= \int_{\mathbb{R}} \left| \widehat{V_N^t}(\boldsymbol{\xi}_2) \right|^2 |g(t)|^2 \left| \frac{dt}{du} \right| dt \\
&\leqslant \left(\sup_{t,\boldsymbol{\xi}_2,\boldsymbol{\xi}_2'} \left| \widehat{V_N^t}(\boldsymbol{\xi}_2) \right|^2 \right) \int_{\mathbb{R}} |g(t)|^2 \left| \frac{dt}{du} \right| dt \\
&\leqslant \frac{b_0^2}{2c_0} \|g(\cdot)\|_{L_t^2}^2 .
\end{aligned}$$

Hence, we find

$$\begin{aligned}
&\left| \int J(f)(t,\boldsymbol{\xi}_1) \overline{g(t)} dt \right| \\
\leqslant\ & \int \frac{|\boldsymbol{\xi}_1|^2 \, d\boldsymbol{\xi}_2'}{|\boldsymbol{\xi}_2'|^2} \int \frac{d\boldsymbol{\xi}_2}{\left| \boldsymbol{\xi}_2 - \frac{\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2'}{2} \right|^2 \left| \boldsymbol{\xi}_2 + \frac{\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2'}{2} \right|^2} \left| \int \widehat{V_N^t}(\boldsymbol{\xi}_2) e^{2i\boldsymbol{\xi}_2^T A_t \boldsymbol{\xi}_2} \left(\overline{e^{-i\phi(t,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2)} g(t)} \right) dt \right| \\
& \left| \int \overline{\widehat{V_N^{t'}}(\boldsymbol{\xi}_2)} e^{-2i\boldsymbol{\xi}_2^T A_{t'} \boldsymbol{\xi}_2} \left(e^{-i\phi(t',\boldsymbol{\xi}_1,\boldsymbol{\xi}_2)} f(t',\boldsymbol{\xi}_1) \right) dt' \right| \\
=\ & \int \frac{|\boldsymbol{\xi}_1|^2 \, d\boldsymbol{\xi}_2'}{|\boldsymbol{\xi}_2'|^2} \int \frac{\left| \overline{\hat{G}(|\boldsymbol{\xi}_2|^2)} \hat{F}(|\boldsymbol{\xi}_2|^2, \boldsymbol{\xi}_1) \right|}{\left| \boldsymbol{\xi}_2 - \frac{\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2'}{2} \right|^2 \left| \boldsymbol{\xi}_2 + \frac{\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2'}{2} \right|^2} d\boldsymbol{\xi}_2 \\
=\ & \int \frac{|\boldsymbol{\xi}_1|^2 \, d\boldsymbol{\xi}_2'}{|\boldsymbol{\xi}_2'|^2} \int \frac{\left| \hat{F}(\rho^2, \boldsymbol{\xi}_1) \overline{\hat{G}(\rho^2)} \right|}{\left| \boldsymbol{\xi}_2 - \frac{\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2'}{2} \right|^2 \left| \boldsymbol{\xi}_2 + \frac{\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2'}{2} \right|^2} \rho^2 d\rho d\boldsymbol{\sigma} \quad \text{(spherical coordinate in } \boldsymbol{\xi}_2\text{)} \\
\leqslant\ & \int \frac{|\boldsymbol{\xi}_1|^2 \, d\boldsymbol{\xi}_2'}{|\boldsymbol{\xi}_2'|^2} \sup_{\rho} \left(\int \frac{\rho^2 d\boldsymbol{\sigma}}{\rho \left| \boldsymbol{\xi}_2 - \frac{\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2'}{2} \right|^2 \left| \boldsymbol{\xi}_2 + \frac{\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2'}{2} \right|^2} \right) \left(\int \left| \hat{F}(\rho^2, \boldsymbol{\xi}_1) \right|^2 \rho d\rho \right)^{\frac{1}{2}} \left(\int \left| \hat{G}(\rho^2) \right|^2 \rho d\rho \right)^{\frac{1}{2}} \\
& \text{(Hölder in } \rho\text{)} \\
\leqslant\ & C b_0^2 \|f(\cdot,\boldsymbol{\xi}_1)\|_{L_t^2} \|g\|_{L_t^2} \left\{ \int \frac{|\boldsymbol{\xi}_1|^2}{|\boldsymbol{\xi}_2'|^2} \sup_{\rho} \left(\int \frac{\rho^2 d\boldsymbol{\sigma}}{\rho \left| \boldsymbol{\xi}_2 - \frac{\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2'}{2} \right|^2 \left| \boldsymbol{\xi}_2 + \frac{\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2'}{2} \right|^2} \right) d\boldsymbol{\xi}_2' \right\}
\end{aligned}$$

However, we have the estimate

$$\begin{aligned}
& \int \frac{|\boldsymbol{\xi}_1|^2}{|\boldsymbol{\xi}_2'|^2} \sup_{\rho} \left(\int \frac{\rho^2 d\boldsymbol{\sigma}}{\rho \left| \boldsymbol{\xi}_2 - \frac{\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2'}{2} \right|^2 \left| \boldsymbol{\xi}_2 + \frac{\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2'}{2} \right|^2} \right) d\boldsymbol{\xi}_2' \\
=\ & \int \frac{|\boldsymbol{\xi}_1|^2}{|\boldsymbol{\xi}_2'|^2} \sup_{\rho} \left(\int \frac{\left| \boldsymbol{\xi}_2 - \frac{\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2'}{2} \right|^2 d\boldsymbol{\sigma}}{\left| \boldsymbol{\xi}_2 - \frac{\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2'}{2} \right| |\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2 - \boldsymbol{\xi}_2'|^2 |\boldsymbol{\xi}_2|^2} \right) d\boldsymbol{\xi}_2' \\
& \text{(Reverse the change of variable in formula 10.2.)} \\
=\ & |\boldsymbol{\xi}_1|^2 \int \frac{d\boldsymbol{\xi}_2'}{|\boldsymbol{\xi}_2'|^{2+2\varepsilon}} \sup_{\rho} \left(\int \frac{\left| \boldsymbol{\xi}_2 - \frac{\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2'}{2} \right|^2 d\boldsymbol{\sigma}}{\left| \boldsymbol{\xi}_2 - \frac{\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2'}{2} \right| |\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2 - \boldsymbol{\xi}_2'|^{2-\varepsilon} |\boldsymbol{\xi}_2|^{2-\varepsilon}} \right) \\
\leqslant\ & C |\boldsymbol{\xi}_1|^2 \int \frac{d\boldsymbol{\xi}_2'}{|\boldsymbol{\xi}_2'|^{2+2\varepsilon} |\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2'|^{3-2\varepsilon}} \quad \text{(Second part of Lemma 12)} \\
\leqslant\ & C.
\end{aligned}$$

In the above calculation, the $\boldsymbol{\sigma}$ in the first line lies on the unit sphere centered at the origin while the $\boldsymbol{\sigma}$ in the second line is on a unit sphere centered at $\frac{\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2'}{2}$. We use the same symbol because Lebesgue measure is translation invariant.

Thus, we conclude that

$$\left|\int J(f)(t,\boldsymbol{\xi}_1)\overline{g(t)}dt\right| \leqslant Cb_0^2 \left\|f(\cdot,\boldsymbol{\xi}_1)\right\|_{L_t^2} \left\|g\right\|_{L_t^2}.$$

$\square$

10.0.6. *Case II: $I(h)$ restricted to the region $\left|\boldsymbol{\xi}_2'\right| > \left|\boldsymbol{\xi}_2\right|$ with integration order $d\boldsymbol{\xi}_2'$ prior to $d\boldsymbol{\xi}_2$.* For this case, we express the phase function as

$$\begin{aligned}
& \left(\boldsymbol{\xi}_1-\boldsymbol{\xi}_2-\boldsymbol{\xi}_2'\right)^T A_t \left(\boldsymbol{\xi}_1-\boldsymbol{\xi}_2-\boldsymbol{\xi}_2'\right) + \boldsymbol{\xi}_2^T A_t \boldsymbol{\xi}_2 - \left(\boldsymbol{\xi}_2'\right)^T A_t \boldsymbol{\xi}_2' \\
= \;& \left(\boldsymbol{\xi}_1-\boldsymbol{\xi}_2\right)^T A_t \left(\boldsymbol{\xi}_1-\boldsymbol{\xi}_2\right) - 2\left(\boldsymbol{\xi}_1-\boldsymbol{\xi}_2\right)^T A_t \boldsymbol{\xi}_2' + \boldsymbol{\xi}_2^T A_t \boldsymbol{\xi}_2 \\
= \;& \phi(t,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2) - 2\left(\boldsymbol{\xi}_1-\boldsymbol{\xi}_2\right)^T A_t \boldsymbol{\xi}_2'.
\end{aligned}$$

and let

$$\begin{aligned}
J\left(\hat{h}\right)(t,\boldsymbol{\xi}_1) \;=\; & \int \frac{\left|\boldsymbol{\xi}_1\right|^2 e^{-2i(\boldsymbol{\xi}_1-\boldsymbol{\xi}_2)^T A_t \boldsymbol{\xi}_2'} e^{2i(\boldsymbol{\xi}_1-\boldsymbol{\xi}_2)^T A_{t'} \boldsymbol{\xi}_2'}}{\left|\boldsymbol{\xi}_1-\boldsymbol{\xi}_2-\boldsymbol{\xi}_2'\right|^2 \left|\boldsymbol{\xi}_2\right|^2 \left|\boldsymbol{\xi}_2'\right|^2} \\
& \widehat{V_N^t}(\boldsymbol{\xi}_2)\overline{\widehat{V_N^{t'}}(\boldsymbol{\xi}_2)} e^{-i\phi(t',\boldsymbol{\xi}_1,\boldsymbol{\xi}_2)} \overline{e^{-i\phi(t,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2')}} \hat{h}(t',\boldsymbol{\xi}_1) dt' d\boldsymbol{\xi}_2' d\boldsymbol{\xi}_2.
\end{aligned}$$

Again, we want to prove the following property.

Proposition 5.

$$\int \left|J(f)(t,\boldsymbol{\xi}_1)\right|^2 dt \leqslant Cb_0^2 \left\|f(\cdot,\boldsymbol{\xi}_1)\right\|_{L_t^2}^2$$

where the constant C is independent of f or $\boldsymbol{\xi}_1$.

Proof. We calculate

$$\begin{aligned}
& \left|\int J(f)(t,\boldsymbol{\xi}_1)\overline{g(t)}dt\right| \\
= \;& \Bigg|\int \frac{\left|\boldsymbol{\xi}_1\right|^2 e^{-2i(\boldsymbol{\xi}_1-\boldsymbol{\xi}_2)^T A_t \boldsymbol{\xi}_2'} e^{2i(\boldsymbol{\xi}_1-\boldsymbol{\xi}_2)^T A_{t'} \boldsymbol{\xi}_2'}}{\left|\boldsymbol{\xi}_1-\boldsymbol{\xi}_2-\boldsymbol{\xi}_2'\right|^2 \left|\boldsymbol{\xi}_2\right|^2 \left|\boldsymbol{\xi}_2'\right|^2} \left(\overline{\widehat{V_N^{t'}}(\boldsymbol{\xi}_2)} e^{-i\phi(t',\boldsymbol{\xi}_1,\boldsymbol{\xi}_2)} f(t',\boldsymbol{\xi}_1)\right) \\
& \left(\widehat{V_N^t}(\boldsymbol{\xi}_2)\overline{e^{-i\phi(t,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2)} g(t)}\right) dt dt' d\boldsymbol{\xi}_2' d\boldsymbol{\xi}_2\Bigg| \\
\leqslant \;& \int \frac{\left|\boldsymbol{\xi}_1\right|^2 d\boldsymbol{\xi}_2}{\left|\boldsymbol{\xi}_2\right|^2} \int \frac{d\boldsymbol{\xi}_2'}{\left|\boldsymbol{\xi}_1-\boldsymbol{\xi}_2-\boldsymbol{\xi}_2'\right|^2 \left|\boldsymbol{\xi}_2'\right|^2} \left|\int e^{-2i(\boldsymbol{\xi}_1-\boldsymbol{\xi}_2)^T A_t \boldsymbol{\xi}_2'} \left(\widehat{V_N^t}(\boldsymbol{\xi}_2)\overline{e^{-i\phi(t,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2)} g(t)}\right) dt\right| \\
& \left|\int e^{2i(\boldsymbol{\xi}_1-\boldsymbol{\xi}_2)^T A_{t'} \boldsymbol{\xi}_2} \left(\overline{\widehat{V_N^{t'}}(\boldsymbol{\xi}_2)} e^{-i\phi(t',\boldsymbol{\xi}_1,\boldsymbol{\xi}_2)} f(t',\boldsymbol{\xi}_1)\right) dt'\right|
\end{aligned}$$

Fix $\boldsymbol{\xi}_1-\boldsymbol{\xi}_2$ and $\boldsymbol{\xi}_2'$, and write

$$\int e^{-2i(\boldsymbol{\xi}_1-\boldsymbol{\xi}_2)^T A_t \boldsymbol{\xi}_2'} \left(\widehat{V_N^t}(\boldsymbol{\xi}_2)\overline{e^{-i\phi(t,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2')} g(t)}\right) dt = \int e^{-2i\left|\boldsymbol{\xi}_1-\boldsymbol{\xi}_2\right|\omega^T A_t \boldsymbol{\xi}_2'} \left(\widehat{V_N^t}(\boldsymbol{\xi}_2)\overline{e^{-i\phi(t,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2')} g(t)}\right) dt$$

where $\boldsymbol{\omega} = (\omega_1,\omega_2,\omega_3)$ is a unit vector in $\mathbb{R}^3$. Without loss of generality, we assume that

$$\max\left\{\left|\omega_1\right|,\left|\omega_2\right|,\left|\omega_3\right|\right\} = \left|\omega_1\right|$$

which in turn implies

$$\frac{1}{\sqrt{3}} \leqslant \left|\omega_1\right| \leqslant 1.$$

We then write

$$\begin{aligned}
\boldsymbol{\xi}_2' &= (x,0,0) + (0,y_1,y_2) \\
u(t) &= 2\int_0^t a_1(s)ds.
\end{aligned}$$

Again u is invertible with

$$\frac{du}{dt} \geqslant 2c_0 > 0.$$

So we have

$$\begin{aligned}
&\int e^{-2i(\boldsymbol{\xi}_1-\boldsymbol{\xi}_2)^T A_t \boldsymbol{\xi}_2'} \left(\widehat{V_N^t}(\boldsymbol{\xi}_2)\overline{e^{-i\phi(t,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2')}g(t)}\right) dt \\
=& \int e^{-2i|\boldsymbol{\xi}_1-\boldsymbol{\xi}_2|\boldsymbol{\omega}^T A_t \boldsymbol{\xi}_2'} \left(\widehat{V_N^t}(\boldsymbol{\xi}_2)\overline{e^{-i\phi(t,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2')}g(t)}\right) dt \\
=& \int e^{-iu(\omega_1|\boldsymbol{\xi}_1-\boldsymbol{\xi}_2|x)} \left(\widehat{V_N^t}(\boldsymbol{\xi}_2)e^{-2i|\boldsymbol{\xi}_1-\boldsymbol{\xi}_2|(0,\omega_2,\omega_3)^T A_{t(u)}(0,y_1,y_2)}\overline{e^{-i\phi(t(u),\boldsymbol{\xi}_1,\boldsymbol{\xi}_2')}g(t(u))}\left|\frac{dt}{du}\right|\right) du \\
=& \overline{\hat{G}(-\omega_1\left|\boldsymbol{\xi}_1-\boldsymbol{\xi}_2\right|x)}
\end{aligned}$$

where

$$G(u)=\overline{\widehat{V_N^t}(\boldsymbol{\xi}_2)e^{-2i|\boldsymbol{\xi}_1-\boldsymbol{\xi}_2|(0,\omega_2,\omega_3)^T A_{t(u)}(0,y_1,y_2)}}e^{-i\phi(t(u),\boldsymbol{\xi}_1,\boldsymbol{\xi}_2')}g(t(u))\left|\frac{dt}{du}\right|.$$

The last expression still has the property that

$$\int |G(u)|^2\,du \leqslant \frac{b_0^2}{2c_0}\int |g(t)|^2\,dt.$$

Just as in case 1, this procedure furnishes

$$\begin{aligned}
&\left|\int J(f)(t,\boldsymbol{\xi}_1)\overline{g(t)}dt\right| \\
\leqslant& \int \frac{|\boldsymbol{\xi}_1|^2\,d\boldsymbol{\xi}_2}{|\boldsymbol{\xi}_2|^2}\int\frac{d\boldsymbol{\xi}_2'}{|\boldsymbol{\xi}_1-\boldsymbol{\xi}_2-\boldsymbol{\xi}_2'|^2\,|\boldsymbol{\xi}_2'|^2}\left|\int e^{-2i(\boldsymbol{\xi}_1-\boldsymbol{\xi}_2)^T A_t\boldsymbol{\xi}_2'}\left(\widehat{V_N^t}(\boldsymbol{\xi}_2)\overline{e^{-i\phi(t,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2)}g(t)}\right)dt\right| \\
&\left|\int e^{2i(\boldsymbol{\xi}_1-\boldsymbol{\xi}_2)^T A_{t'}\boldsymbol{\xi}_2}\left(\overline{\widehat{V_N^{t'}}(\boldsymbol{\xi}_2)}e^{-i\phi(t',\boldsymbol{\xi}_1,\boldsymbol{\xi}_2)}f(t',\boldsymbol{\xi}_1)\right)dt'\right| \\
=& \int\left(\int\frac{dx\,dy_1\,dy_2}{|\boldsymbol{\xi}_1-\boldsymbol{\xi}_2-\boldsymbol{\xi}_2'|^2\,|\boldsymbol{\xi}_2'|^2}\left|\overline{\hat{G}(-\omega_1\left|\boldsymbol{\xi}_1-\boldsymbol{\xi}_2\right|x)}\hat{F}(-\omega_1\left|\boldsymbol{\xi}_1-\boldsymbol{\xi}_2\right|x,\boldsymbol{\xi}_1)\right|\right)\frac{|\boldsymbol{\xi}_1|^2}{|\boldsymbol{\xi}_2|^2}d\boldsymbol{\xi}_2 \\
=& \int\left(\int\frac{dx\,dy_1\,dy_2}{|\boldsymbol{\xi}_1-\boldsymbol{\xi}_2-\boldsymbol{\xi}_2'|^2\,|\boldsymbol{\xi}_2'|^2}\left|\overline{\hat{G}(x)}\hat{F}(x,\boldsymbol{\xi}_1)\right|\right)\frac{|\boldsymbol{\xi}_1|^2}{|\omega_1|\,|\boldsymbol{\xi}_1-\boldsymbol{\xi}_2|\,|\boldsymbol{\xi}_2|^2}d\boldsymbol{\xi}_2 \\
\leqslant& C\sqrt{3}\int\frac{|\boldsymbol{\xi}_1|^2}{|\boldsymbol{\xi}_1-\boldsymbol{\xi}_2|\,|\boldsymbol{\xi}_2|^2}\left(\sup_x\int\frac{dy_1\,dy_2}{|\boldsymbol{\xi}_1-\boldsymbol{\xi}_2-\boldsymbol{\xi}_2'|^2\,|\boldsymbol{\xi}_2'|^2}\right) \\
&\left(\int\left|\hat{F}(x,\boldsymbol{\xi}_1)\right|^2dx\right)^{\frac{1}{2}}\left(\int\left|\hat{G}(x)\right|^2dx\right)^{\frac{1}{2}}d\boldsymbol{\xi}_2 \text{ (Hölder in } x) \\
\leqslant& Cb_0^2\left\|f(\cdot,\boldsymbol{\xi}_1)\right\|_{L_t^2}\|g\|_{L_t^2}\int\frac{|\boldsymbol{\xi}_1|^2}{2\,|\boldsymbol{\xi}_1-\boldsymbol{\xi}_2|\,|\boldsymbol{\xi}_2|^2}\left(\sup_x\int\frac{dy_1\,dy_2}{|\boldsymbol{\xi}_1-\boldsymbol{\xi}_2-\boldsymbol{\xi}_2'|^2\,|\boldsymbol{\xi}_2'|^2}\right)d\boldsymbol{\xi}_2.
\end{aligned}$$

The first part of Lemma 12 along with the restrictions $\left|\boldsymbol{\xi}_1-\boldsymbol{\xi}_2-\boldsymbol{\xi}_2'\right|>|\boldsymbol{\xi}_2|$ and $\left|\boldsymbol{\xi}_2'\right|<|\boldsymbol{\xi}_2|$ entail

$$\begin{aligned}
&\int\frac{|\boldsymbol{\xi}_1|^2}{2\,|\boldsymbol{\xi}_1-\boldsymbol{\xi}_2|\,|\boldsymbol{\xi}_2|^2}\left(\sup_x\int\frac{dy_1\,dy_2}{|\boldsymbol{\xi}_1-\boldsymbol{\xi}_2-\boldsymbol{\xi}_2'|^2\,|\boldsymbol{\xi}_2'|^2}\right)d\boldsymbol{\xi}_2 \\
\leqslant& \int\frac{|\boldsymbol{\xi}_1|^2}{2\,|\boldsymbol{\xi}_1-\boldsymbol{\xi}_2|\,|\boldsymbol{\xi}_2|^{2+2\varepsilon}}\left(\sup_x\int\frac{dy_1\,dy_2}{|\boldsymbol{\xi}_1-\boldsymbol{\xi}_2-\boldsymbol{\xi}_2'|^{2-\varepsilon}\,|\boldsymbol{\xi}_2'|^{2-\varepsilon}}\right)d\boldsymbol{\xi}_2 \\
\leqslant& C\int\frac{|\boldsymbol{\xi}_1|^2\,d\boldsymbol{\xi}_2}{2\,|\boldsymbol{\xi}_1-\boldsymbol{\xi}_2|^{3-2\varepsilon}\,|\boldsymbol{\xi}_2|^{2+2\varepsilon}} \\
\leqslant& C,
\end{aligned}$$

which finishes the proposition. □

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